SOLVING LINEAR EQUATIONS

NOTATION: The *leading entry* of a row of a matrix is the left-most non-zero entry of that row.
A *pivot position* of the matrix A is a position in A that corresponds to a leading entry 1 in a row of *the Reduced Row Echelon matrix for A.*
A *pivot column* is a column of A that contains a pivot position.

One method to turn a matrix A into a reduced row echelon (RRE) matrix using row operations:

Step 1. Be clever. Then, use *interchange operations* to put all rows consisting entirely of zeros at the bottom of the matrix.
Step 2. Now, begin with the leftmost nonzero column. This is a pivot column.
Step 3. Choose a simple row with a nonzero entry in this pivot column. If necessary, use *interchange operations* to move this row to the first row.
Step 4. *Clean down* this pivot column! That is, use row *replacement operations* to create zeros in all positions in that column below this pivot.
Step 5. Now, forget about this row. Apply steps 1-3 to the submatrix below this row.
Repeat this procedure and keep cleaning down successive columns.

$\star$ *At this point, the matrix is a row echelon matrix (but not necessarily in RRE).* FYI, the leading entries of this reduced matrix are in the pivot positions for A.

Step 6. *Clean up.*
- (a) Start with the rightmost pivot and use *replacement operations* and this row to create zeros in this column above this pivot. If the pivot is not 1 make it 1 by a *scaling operation*.
- (b) Consider the row just above this last row. Its pivot is to the left of the pivot in the lowest row. Use *replacement operations* and this row to create zeros above this pivot. If the pivot is not 1 make it 1 by a *scaling operation*.
- (c) Continue cleaning up. Then, make sure all leading entries are equal to one and the matrix is in RRE.

To solve $m \times n$ linear algebraic equations $A\mathbf{x} = \mathbf{b}$:

1. Form the augmented matrix $[A\,\mathbf{b}]$ and reduce to RRE.
   - (a) If in this process you get a leading entry in the last column (the matrix has a row: $(0, \ldots, 0, b)$ where $b \neq 0$) then the system is inconsistent. (That is, there are no solutions to the equations because one of the equations is $0 = b$.)
   - (b) If not, continue to reduce to RRE, convert to equations and solve for the *basic variables*[1] (variables corresponding to leading entries of the reduced matrix) in terms of the *free variables* (non-basic variables) and constants. Note that the basic (leading) variables correspond to pivot columns of A. Often one then puts the solution into parametric vector form.

To solve the $m \times n$ homogeneous linear algebraic equations $A\mathbf{x} = \mathbf{0}$:

1. Reduce the augmented matrix $[A\,\mathbf{0}]$ to RRE. (You can just reduce A as long as you convert back to *homogeneous* equations.)
2. Solve for the leading variables in terms of free variables (if there are free variables).
3. If there are free variables: successively set each free variable equal to one (or an 'easy' non-zero number) and set the other free variables equal to zero. Then, write your solutions as vectors. This gives a list of *linearly independent* solution vectors $\mathbf{u}_1, \ldots, \mathbf{u}_m$(one for each free variable). Every solution to $A\mathbf{x} = \mathbf{0}$ is a linear combination of these solution vectors; therefore the solution set to $A\mathbf{x} = \mathbf{0}$ is $\text{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$.

[1] I (and many books) refer to basic variables as *leading variables*.