MATH 70 FINAL F2005 Selected Solns

[1] $A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$

a) Can you solve $Ax=b$? $\left[\begin{array}{cc|c} 1 & 3 & 5 \\ 1 & -1 & 1 \\ 1 & 1 & 0 \end{array}\right] \begin{array}{l} \\ -R_1 \\ -R_1 \end{array} \left[\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & -4 & -4 \\ 0 & -2 & -5 \end{array}\right] \left(-\tfrac{1}{4}\right) \left[\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & 1 & 1 \\ 0 & -2 & -5 \end{array}\right] +2R2 \left[\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & -3 \end{array}\right]$ No

Inconsistent

b) OPTION 1 NORMAL EQS: From $Ax=b$

TO $\boxed{A^TA\hat{x} = A^Tb}$

$A^TA = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 11 \end{bmatrix}$ $\quad A^Tb = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \end{bmatrix}$

Solve $\begin{bmatrix} 3 & 3 \\ 3 & 11 \end{bmatrix} \hat{x} = \begin{bmatrix} 6 \\ 14 \end{bmatrix}$:

$\left[\begin{array}{cc|c} 3 & 3 & 6 \\ 3 & 11 & 14 \end{array}\right] (1/3) \quad \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 3 & 11 & 14 \end{array}\right] -3R1 \quad \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 8 & 8 \end{array}\right] \left(\tfrac{1}{8}\right) \quad \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 1 \end{array}\right] -R2$

$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right]$ $\boxed{\hat{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$

OPTION 2 ① Find an OG basis for Col A. If $A = [v_1 \ v_2]$ Let $A' = [x_1 \ x_2]$ $\quad x_1 = v_1$

$x_2 = v_2 - \mathrm{proj}_{x_1} v_2 = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} - \frac{3}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix}$ So $A' = \begin{pmatrix} 1 & 2 \\ 1 & -2 \\ 1 & 0 \end{pmatrix}$ $\quad \boxed{b = \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix}}$

② Compute $\hat{b} = \mathrm{proj}_{\mathrm{Col}\,A'} b = \mathrm{proj}_{x_1} b + \mathrm{proj}_{x_2} b = \frac{6}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{8}{8}\begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix} = \boxed{\begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix}}$ $\hat{b}$

③ Solve $A\hat{x} = \hat{b}$ for $\hat{x}$ $\left[\begin{array}{cc|c} 1 & 3 & 4 \\ 1 & -1 & 0 \\ 1 & 1 & 2 \end{array}\right] \begin{array}{l} \\ -R_1 \\ -R_1 \end{array} \sim \left[\begin{array}{cc|c} 1 & 3 & 4 \\ 0 & -4 & -4 \\ 0 & -2 & -2 \end{array}\right] \left(-\tfrac{1}{4}\right) \sim \left[\begin{array}{cc|c} 1 & 3 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right] -3R2 \quad \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$ $\hat{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ ☆

$\boxed{1}$ c) LS error $= \|b - \hat{b}\| = \left\| \begin{pmatrix}5\\1\\0\end{pmatrix} - \begin{pmatrix}4\\0\\2\end{pmatrix} \right\| = \left\| \begin{pmatrix}1\\1\\-2\end{pmatrix} \right\|$

$= \sqrt{1^2 + 1^2 + (-2)^2} = \sqrt{6}$

$\boxed{3}$ Let $T: \mathbb{P}_2 \to \mathbb{R}^3$ be the linear transformation given by $T(p(t)) = \begin{pmatrix} p(-1) \\ p(0) \\ p'(-1/2) \end{pmatrix}$

So $T(a + bt + ct^2) = \begin{pmatrix} a - b + c \\ a \\ b + 2(c)(-\frac{1}{2}) \end{pmatrix} = \begin{pmatrix} a - b + c \\ a \\ b - c \end{pmatrix}$

$p'(t) = b + 2ct$

a) $\ker T$ is a subspace of the domain $\mathbb{P}_2$

b) Find $\ker T$: $\ker T = \left\{ a + bt + ct^2 \;\middle|\; \begin{matrix} a - b + c = 0 \\ a = 0 \\ b - c = 0 \end{matrix} \right\}$

$= \{ a + bt + ct^2 \mid a = 0 \text{ and } b = c \}$

$= \{ bt + bt^2 : b \in \mathbb{R} \}$. So a basis for $\ker T = \{ t + t^2 \}$

c) [OMIT c's — this is from §4.4]

$\boxed{5}$ $p_1 = 1 + t^2$, $p_2 = t$, $p_3 = 1 + t$

a) verify that $\{p_1, p_2, p_3\}$ is a basis for $\mathbb{P}_2$

<u>option 1:</u> $p_1 \neq 0_{\mathbb{P}_2}$, p_2 is not a multiple of p_1; p_3 is not a linear combination of p_1 and p_2 because it would have to be $p_1 + p_2 = 1 + t + t^2 \neq 1 + t$.

Since $\{p_1, p_2, p_3\}$ is LI and since $\dim \mathbb{P}_2 = 3$, $\{p_1, p_2, p_3\}$ is a basis for $\mathbb{P}_2$

<u>option 2</u> Let $\mathcal{B}' = \{1, t, t^2\}$ be the standard basis for $\mathbb{P}_2$.

$\mathcal{B} = \{p_1, p_2, p_3\}$ is a basis for $\mathbb{P}_2$ if and only if $\{[p_1]_{\mathcal{B}'}, [p_2]_{\mathcal{B}'}, [p_3]_{\mathcal{B}'}\}$ is a basis for $\mathbb{R}^3$:

$$\underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}}_{M} \begin{matrix} \\ \\ -R1 \end{matrix} \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

Cols are LI and $\dim \mathbb{R}^3 = 3 \Rightarrow$ the cols of M form a basis for $\mathbb{R}^3 \Rightarrow$ $\mathcal{B}$ <u>is</u> a basis for $\mathbb{P}_2$