A few extra proof problems for the final

1. Let V and W be vector spaces and let $T : V \to W$ be linear. Prove that T is one-to-one if and only if $\ker(T) = \{\mathbf{0}\}$.

2. Let A be an $m \times n$ matrix. Prove that the columns of A are independent if and only if $\text{Nul}(A) = \{\mathbf{0}\}$.

3. Let V and W be vector spaces and let $T : V \to W$ be linear. Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ span V.
   (a) Prove that $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_k)\}$ spans range(T).
   (b) Now assume T is onto W. Prove that $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_k)\}$ spans W.

4. Describe the zero vectors in the following vector spaces: $\mathbb{R}^n$, $\mathbb{P}_n$, $M_{2\times 3}$.