A few extra proof problems for the final

1. Let V and W be vector spaces and let $T : V \to W$ be linear. Prove that T is one-to-one if and only if $\ker(T) = \{\mathbf{0}\}$.

Answer: *First you should always write down (or at least make sure you know) the definitions of all of the terms in the problem.*

Next, you should write down the hypotheses (assumptions) and conclusions (goals of the problem). In an if and only if (IFF) problem, you should prove both directions.

So, first we will assume T is one-to-one. This means that if $\mathbf{u}$ and $\mathbf{v}$ are in V and $T(\mathbf{u}) = T(\mathbf{v})$ then $\mathbf{u} = \mathbf{v}$.

Goal: *we want to show that* $\ker(T) = \{\mathbf{0}\}$.

Recall the definition of $\ker(T) = \left\{\mathbf{v} \in V \,\middle|\, T(\mathbf{v}) = \mathbf{0}\right\}$.

Let $\mathbf{v} \in \ker(T)$, then $T(\mathbf{v}) = \mathbf{0}$. However, because T is linear, $T(\mathbf{0}) = \mathbf{0}$. As T is one-to-one, since $T(\mathbf{v}) = \mathbf{0} = T(\mathbf{0})$, $\mathbf{v} = \mathbf{0}$ and $\ker(T) = \{\mathbf{0}\}$, the set consisting only of the zero element of V.

Now, we assume $\ker(T) = \{\mathbf{0}\}$.

Goal: *We want to show T is one-to-one.*

To check the definition, we let $\mathbf{u}$ and $\mathbf{v}$ be vectors in V and we assume $T(\mathbf{u}) = T(\mathbf{v})$.

(We need to use the assumption that $\ker(T) = \{\mathbf{0}\}$ so we rewrite $T(\mathbf{u}) = T(\mathbf{v})$ to become

$$\mathbf{0} = T(\mathbf{u}) - T(\mathbf{v}) = T(\mathbf{u} - \mathbf{v}) \quad \textit{since } T \textit{ is linear.}$$

Since $\ker(T) = \{\mathbf{0}\}$, $\mathbf{u} - \mathbf{v} = \mathbf{0}$ so $\mathbf{u} = \mathbf{v}$. This finishes the proof.

2. Let A be an $m \times n$ matrix. Prove that the columns of A are independent if and only if $\mathrm{Nul}(A) = \{\mathbf{0}\}$.

Answer: *To talk about the columns of A, we first need to label them, so let $A = [\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n]$ where $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are all in $\mathbb{R}^m$.*

In this proof, we will prove both directions at the same time by making each step an IFF proof, thereby proving both directions at the same time.

The columns of A are independent IFF the only weights $x_1, \ldots, x_n$ that make

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{0} \tag{1}$$

are $x_1 = x_2 = \cdots = x_n = 0$ by the definition of linear independence.

Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, *then the vector equation (1) is equivalent to the matrix equation $A\mathbf{x} = \mathbf{0}$.*

Therefore, the columns of A are linearly independent IFF the matrix equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \mathbf{0}$.

Since $\mathrm{Nul}(A)$ is the set of all solutions to $A\mathbf{x} = \mathbf{0}$, this means that the columns of A are linearly independent IFF $\mathrm{Nul}(A) = \{\mathbf{0}\}$, and this proves the statement.

3. Let V and W be vector spaces and let $T : V \to W$ be linear. Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ span V.

(a) Prove that $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_k)\}$ spans range(T).

Answer: *Before starting, we think of the definition of* range(T) *and spanning set. Then we begin!*
Let $\mathbf{w} \in \text{range}(T)$*. Then, by definition* range(T)*, there is a* $\mathbf{v} \in V$ *such that* $T(\mathbf{v}) = \mathbf{w}$*.*
(Now relate this fact to the assumption that $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ spans V.)
Since $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ *spans* V*, there are weights* $c_1, \ldots, c_k$ *in* $\mathbb{R}$ *such that* $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$*.*
(Now relate to $T(\mathbf{v}) = \mathbf{w}$.)
Then, $\mathbf{w} = T(\mathbf{v}) = T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k)$*,*
By linearity we see $\mathbf{w} = T(\mathbf{v}) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + \cdots + c_kT(\mathbf{v}_k)$*, and, since* $\mathbf{w}$ *is an arbitrary element of* range(T)*,* $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_k)\}$ *spans* range(T)*.*

(b) Now assume T maps onto W. Prove that $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_k)\}$ spans W.

Answer: *Since* T *is onto* W*, this means* $W = \text{range}(T)$*, and by the result of part (a),* $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_k)\}$ *spans* W*.*

4. Describe the zero vectors in the following vector spaces: $\mathbb{R}^n$, $\mathbb{P}_n$, $M_{2\times 3}$.

Answer: *The point of this question is to help you realize that* $\mathbf{0}$ *means different things in different vector spaces. So:*

In $\mathbb{R}^n$*,* $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$.

In P_n*,* $\mathbf{0}$ *is the zero polynomial so* $\forall t \in \mathbb{R}$*,* $\mathbf{0}(t) = 0$ *(for all* $t \in \mathbb{R}$*, the zero polynomial evaluated at* t *is equal to the number zero).*

In $M_{2\times 3}$ $\mathbf{0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$*, the* 2×3 *matrix with all entries being zero.*

In each of these vector spaces, these vectors satisfy $\mathbf{0} + \mathbf{v} = \mathbf{v} + \mathbf{0} = \mathbf{v}$ *for all* $\mathbf{v}$ *in the vector space, which is the condition to be the zero vector.*