Math 70	**TUFTS UNIVERSITY**	**May 9, 2016, 8:30-10:30 A.M.**
Linear Algebra	**Final Exam**	**Department of Mathematics**

Instructions: No notes or books are allowed. All calculators, cell phones, or other electronic devices **must** be turned off and put away during the exam. Unless otherwise stated, you **must show all work** to receive full credit. *You are required to sign the last page of your exam. With your signature you are pledging that you have neither given nor received assistance on the exam. Students found violating this pledge will receive an F in the course.*

Problem	Point Value	Points
1	10	
2	2	
3	6	
4	8	
5	10	
6	8	
7	10	
8	8	
9	8	
10	6	
11	8	
12	8	
13	8	
	100	

1. (10 points) For each question, indicate your answer by shading the appropriate box. No partial credit.

   (a) $\mathbb{R}^2$ is a subspace of $\mathbb{R}^3$ $\boxed{\text{T}}$ $\boxed{\text{F}}$

   (b) $\mathbb{P}_2$ is a subspace of $\mathbb{P}_3$ ($\mathbb{P}_n$ is the set of polynomials of degree less than or equal to n). $\boxed{\text{T}}$ $\boxed{\text{F}}$

   (c) Is it possible to have a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ with the property that $T(u) = T(v)$ for some pair of distinct vectors u and v in $\mathbb{R}^n$ *and* that T is onto $\mathbb{R}^n$? $\boxed{\text{YES}}$ $\boxed{\text{NO}}$

   (d) Every orthogonal set in $\mathbb{R}^n$ has at most n vectors in it. $\boxed{\text{T}}$ $\boxed{\text{F}}$

   (e) If the orthogonal projection of a vector $\mathbf{v}$ onto a subspace W equals $\mathbf{v}$, then $\mathbf{v} \in W$. $\boxed{\text{T}}$ $\boxed{\text{F}}$

2. (2 points) Let V be a vector space. Consider the three sets
   i. S_1 is a linearly independent subset of V but it does not span V;
   ii. S_2 is a spanning set of V but it is not linearly independent, and
   iii. S_3 is a basis of V.

   Order the sets from smallest to largest in the spaces below.

   __________ __________ __________

3. (6 points) Let A be an $n \times n$ matrix such that $\det(A^4) = 0$. Is A invertible? Justify your answer.

4. (8 points) Let $A = \begin{bmatrix} 4 & 1 \\ 3 & 6 \end{bmatrix}$.

   (a) Find all eigenvalues of A.

   (b) Show that A is diagonalizable by finding an invertible matrix P and diagonal matrix D such that $A = PDP^{-1}$

5. (10 points) Suppose A is a 4×4 matrix *and assume* $\lambda = 0$ *is an eigenvalue of* A.

   (a) Define what it means that $\lambda = 0$ is an eigenvalue of A.

   (b) Use the assumption that $\lambda = 0$ is an eigenvalue and the definition of linear dependence to prove that the columns of A are linearly dependent.

   (c) The maximum rank of A (dimension of Col A) is ______.

6. (8 points) Let $A = \begin{bmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{bmatrix}$. The characteristic polynomial of A is $p(\lambda) = (\lambda - 1)(\lambda + 3)^2$.

(a) Find a basis for the eigenspace corresponding to $\lambda = -3$.

(b) Is A diagonalizable? Justify your answer.

7. (10 points) Define the transformation $T : \mathbb{P}_2 \to \mathbb{M}_{2\times 2}$ by $T(a + bt + ct^2) = \begin{bmatrix} a+2b & b-c \\ 5c & 0 \end{bmatrix}$.

Then T is linear. You do not need to show this.

Let $\mathcal{B} = \{1, t, t^2\}$ and $\mathcal{C} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

be bases for $\mathbb{P}_2$ and $\mathbb{M}_{2\times 2}$, respectively. Find each of the following:

(a) $T(4t + 5t^2)$

(b) The kernel of T.

(c) The matrix for T relative to the bases $\mathcal{B}$ and $\mathcal{C}$. (Referred to as ${}_{\mathcal{C}}[T]_{\mathcal{B}}$ or ${}_{\mathcal{C}}M_{\mathcal{B}}$.)

8. (8 points) Let $T : \mathbb{P}_2 \to W$ be a linear transformation.

Let $\mathcal{B} = \{1, t,\, t^2\}$ and $\mathcal{C} = \{e^x, \cos(x), \sin(x)\}$ be bases for $\mathbb{P}_2$ and W, respectively.

Let $M = {}_{\mathcal{C}}[T]_{\mathcal{B}} = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 1 \\ -1 & 1 & 3 \end{bmatrix}$ be the matrix of the transformation relative to the bases $\mathcal{B}$ and $\mathcal{C}$.

(a) Find $[4 - 3t + t^2]_{\mathcal{B}}$

(b) Find $T(4 - 3t + t^2)$.

9. (8 points) Let $\mathbf{w}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $\mathbf{w}_2 = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$ and let $\mathbf{b} = \begin{bmatrix} 4 \\ 0 \\ 8 \end{bmatrix}$.

(a) Show that $\mathbf{w}_1$ and $\mathbf{w}_2$ are orthogonal.

(b) Find the distance from $\mathbf{b}$ to $W = \mathrm{Span}\{\mathbf{w}_1, \mathbf{w}_2\}$.

Recall: $\mathbf{w}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $\mathbf{w}_2 = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$ and let $\mathbf{b} = \begin{bmatrix} 4 \\ 0 \\ 8 \end{bmatrix}$.

(c) Let A be the matrix $A = [\mathbf{w}_1\ \mathbf{w}_2]$. Decide whether $A\mathbf{x} = \mathbf{b}$ is consistent and explain your answer.

(d) Find all least-squares solutions to $A\mathbf{x} = \mathbf{b}$.

10. (6 points) Let $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix}$, $\mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 8 \\ 8 \end{bmatrix}$. Use the Gram-Schmidt process to find an orthogonal basis of $W = \mathrm{Span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$. You may assume that $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is a basis of W.

11. (8 points) Let $\mathbf{w}_1$ and $\mathbf{w}_2$ be vectors in $\mathbb{R}^3$. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $T(\mathbf{v}) = \begin{bmatrix} \mathbf{v} \cdot \mathbf{w}_1 \\ \mathbf{v} \cdot \mathbf{w}_2 \end{bmatrix}$.

Prove that T is linear.

12. (8 points) Let V and W be vector spaces and let $T : V \to W$ be a linear transformation *that is one-to-one*. Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a linearly independent set of a vectors in V. Prove that the set $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ is linearly independent in W.

13. (8 points) Let W be a subspace of $\mathbb{R}^n$.
Its orthogonal complement is $W^\perp = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \cdot \mathbf{w} = 0 \text{ for all } \mathbf{w} \in W\}$.
Use the definition of subspace to prove that $W^\perp$ is a subspace of $\mathbb{R}^n$.

End of Test. Please fill in the information on the next page.
Have a great summer!

Math 70 **Final Exam** **May 9, 2016**

Name: ______________________________

Circle the name of your instructor

Jessica Dyer

Mary Glaser   Glaser's class: 4-digit secret code which I will use to post grades: ________

Hao Liang

Todd Quinto

I pledge that I have neither given nor received assistance on this exam.

Signature ______________________________