KEY

Math 70	**TUFTS UNIVERSITY**	**May 3, 2013**
Linear Algebra	**Department of Mathematics**	**All sections**
	Final Exam	

Instructions: No notes or books are allowed. All calculators, cell phones, or other electronic devices **must** be turned off and put away during the exam. Unless otherwise stated, you **must show all work** to receive full credit. *You are required to sign your exam. With your signature you are pledging that you have neither given nor received assistance on the exam. Students found violating this pledge will receive an F in the course.*

Problem	Point Value	Points
1	10	
2	12	
3	14	
4	5	
5	6	
6	6	
7	8	
8	8	
9	12	
10	6	
11	8	
12	5	
	100	

1. (10 pts) **True/false questions.** For each of the statements below, decide whether it is true or false. Indicate your answer by shading the corresponding box. There will be no partial credit.

   (a) Let W be a subspace of $\mathbb{R}^n$. W and $W^\perp$ have no vector in common. T F

   $0 \in W \cap W^\perp$

   (b) If $A \in M_{n\times n}$ is similar to a diagonal matrix, then A has n distinct eigenvalues. T F

   I

   F

   (c) The zero vector is contained in any eigenspace and is hence an eigenvector. T F

   (d) Similar matrices have the same eigenvalues. T F

   (e) It is possible for $A \in \mathbb{M}_{5\times 5}$ to have 5 complex eigenvalues. T F

2. (12 points) Set $\mathbf{w}_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{w}_2 = \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$; then $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{v}\}$ is a basis of $\mathbb{R}^3$ (you don't have to verify this). Let W be the plane in $\mathbb{R}^3$ spanned by $\{\mathbf{w}_1, \mathbf{w}_2\}$.

(a) Verify that $\{\mathbf{w}_1, \mathbf{w}_2\}$ is an orthogonal set.

$w_1 \cdot w_2 = -2 + 4 - 2 = 0$ ✓

(b) Verify that $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{v}\}$ is *not* an orthogonal set.

$w_1 \cdot v = 2 + 0 + 0 = 2 \neq 0$

(c) Find a vector $\hat{\mathbf{v}}$ in W and a vector $\mathbf{x}$ in $W^\perp$ such that $\mathbf{v} = \hat{\mathbf{v}} + \mathbf{x}$.

$$\hat{v} = \frac{w_1 \cdot v}{w_1 \cdot w_1} w_1 + \frac{w_2 \cdot v}{w_2 \cdot w_2} w_2 = \frac{2}{9}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + \frac{-1}{9}\begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 5/9 \\ 2/9 \\ 4/9 \end{bmatrix}$$

$$x = v - \hat{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 5/9 \\ 2/9 \\ 4/9 \end{bmatrix} = \begin{bmatrix} 4/9 \\ -2/9 \\ -4/9 \end{bmatrix}$$

(d) Find a vector $\mathbf{w}_3$ such that $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is an orthogonal basis of $\mathbb{R}^3$.

$$w_3 = x = \begin{bmatrix} 4/9 \\ -2/9 \\ -4/9 \end{bmatrix}$$

3. (14 points) Let $A = \begin{bmatrix} 2 & 0 & 1 & 2 \\ 1 & -1 & 0 & 0 \\ 0 & 2 & 1 & 2 \end{bmatrix}$.

(a) Find the solution set to the matrix equation $Ax = 0$. Express your answer in **vector parametric form.**

$$A \sim \begin{bmatrix} 1 & -1 & 0 & 0 \\ 2 & 0 & 1 & 2 \\ 0 & 2 & 1 & 2 \end{bmatrix} -2R_1 \quad \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 2 & 1 & 2 \\ 0 & 2 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tfrac{1}{2}$$

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 1/2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} +R_2 \quad \begin{bmatrix} 1 & 0 & 1/2 & 1 \\ 0 & 1 & 1/2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = -\frac{1}{2}x_3 - x_4$
$x_2 = -\frac{1}{2}x_3 - x_4$
$x_3 = x_3$
$x_4 = x_4$

$$\vec{x} = s\begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

(b) Find a basis for $\text{Nul}(A)$.

$$\mathcal{B} = \left\{ \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ or } \left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} \right\} \text{ etc}$$

(c) Give the dimension of $\text{Col}(A)$ (justify your answer).

# of cols of A = dim Nul A + dim Col A

$4 = 2 + \dim \text{Col } A \Rightarrow \dim \text{Col } A = 2$

(d) Do the columns of A span $\mathbb{R}^3$? Why or why not.

No $\dim \text{Col } A = 2 < \dim \mathbb{R}^3 = 3$

(e) Consider the linear transformation $T(x) = Ax$. Is T onto? Why or why not.

No dim range of T = dim Col A = 2 < dim codomain = dim $\mathbb{R}^3$ = 3

(f) Is T one-to-one? Why or why not.

No. Nul A = ker T $\neq \mathbf{0}$ so T not 1-1

4. (5 points) Using any appropriate method, find the inverse of the matrix $A = \begin{bmatrix} 5 & -2 \\ -3 & 4 \end{bmatrix}$.

$A^{-1} = \frac{1}{14}\begin{bmatrix} 4 & 2 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 4/14 & 2/14 \\ 3/14 & 5/14 \end{bmatrix}$

5. (6 points) Given that A, B, C are $n \times n$ matrices with $\det(A) = -1$, $\det(B) = 2$, and $\det(C) = 4$, find the following:

(a) $\det(ABC)$ $= |A||B||C| = -8$

(b) $\det(B^T C^T)$ $= |B||C| = 8$

(c) $\det(C^{-1}B)$ $= \frac{1}{|C|}|B| = \frac{2}{4} = \frac{1}{2}$

6. (6 points) Let $V = \mathbb{M}_{2\times 2}$ be the set of all 2×2 matrices. Let H be the subset of all matrices in V which have rank less than or equal to 1. Is H a subspace of V? If your answer is "yes", give a proof of your assertion. If your answer is "no", give a counterexample that shows H cannot be a subspace of V.

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ dim rank $A = 1$

$B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ dim rank $B = 1$

$A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ dim rank$(A+B) = 2$ $\ddot{\frown}$

$\therefore$ H is not a subspace - not closed under $+$

7. (8 points) Let u_1, u_2, u_3 be 3 linearly independent vectors in $\mathbb{R}^5$, and let $H = \text{span}\{u_1, u_2, u_3\}$. Let $W = \text{span}\{u_1, u_2, u_1 + u_2 + u_3\}$. Show that $W = H$ by completing the following.

(a) Show that an arbitrary element in H must be an element of W.

Let $w \in H$. Then $\exists$ scalars c_1, c_2, c_3 st $w = c_1u_1 + c_2u_2 + c_3u_3$

$= c_1u_1 + c_2u_2 + c_3[(u_1 + u_2 + u_3)] - c_3u_1 - c_3u_2 \in W$.

$\therefore H \subseteq W$

(b) Show that an arbitrary element of W must be an element of H.

Let $v \in H$. $\exists$ scalars d_1, d_2, d_3 st $v = d_1u_1 + d_2u_2 + d_3(u_1 + u_2 + u_3)$

$= d_1u_1 + d_2u_2 + d_3u_1 + d_3u_2 + d_3u_3$

$= (d_1 + d_3)u_1 + (d_2 + d_3)u_2 + d_3u_3 \in W$.

$\therefore H \subseteq W$

$\therefore H = W$

8. (8 points) Let $A = \begin{array}{cc} a_1 & a_2 \\ \end{array}$ $\begin{bmatrix} 1 & 2 \\ -1 & 4 \\ 1 & 2 \end{bmatrix}$ and $b = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}$.

(a) Verify that the system $Ax = \mathbf{b}$ is inconsistent.

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ -1 & 4 & -1 \\ 1 & 2 & 5 \end{array}\right] \begin{array}{l} \\ +R_1 \\ -R_1 \end{array} \quad \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 6 & 2 \\ 0 & 0 & 2 \end{array}\right] \quad ☹$$

(b) Compute the least squares solution to $Ax = \mathbf{b}$.

Cols of A are $\perp$!!

$$\hat{x} = \begin{bmatrix} \frac{a_1 \cdot b}{a_1 \cdot a_1} \\ \frac{a_2 \cdot b}{a_2 \cdot a_2} \end{bmatrix} = \begin{bmatrix} \frac{9}{3} \\ \frac{12}{24} \end{bmatrix} = \begin{bmatrix} 3 \\ \frac{1}{2} \end{bmatrix}$$

$A^T A \tilde{x} = A^T b$

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 4 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 24 \end{bmatrix}$$

$A^T b \quad \begin{bmatrix} 1 & -1 & 1 \\ 2 & 4 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 9 \\ 12 \end{bmatrix}$

$$\left[\begin{array}{cc|c} 3 & 0 & 9 \\ 0 & 24 & 12 \end{array}\right] \begin{array}{l} 1/3 \\ 1/24 \end{array} \quad \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 1/2 \end{array}\right] \checkmark$$

$$\tilde{x} = \begin{pmatrix} 3 \\ 1/2 \end{pmatrix}$$

9. (12 points) Let V and W be vector spaces and let $T : V \to W$ be a linear transformation.
(a) Using set notation, give the definition of ker(T).

$\ker T = \{ v \in V \mid T(v) = \theta_W \}$

(b) Prove that ker(T) is a subspace of V.

① $\theta_V \in \ker T$ since T linear $\Rightarrow T(\theta_V) = \theta_W$ ✓

② let $u, v \in \ker T$

Then $\underset{\substack{\uparrow \\ T \text{ linear}}}{T(u+v) =} \underset{\substack{\uparrow \\ u,v \in \ker T}}{T(u) + T(v) =} \theta_W + \theta_W = \theta_W$

so $\ker T$ closed under +.

③ let $u \in \ker T$, c a scalar

$\underset{\substack{\uparrow \\ T \text{ lin}}}{T(cu) =} \underset{\substack{\uparrow \\ u \in \ker T}}{cT(u) =} c \cdot \theta_W = \theta_W$ so $\ker T$ closed under scalar mult

10. (6 points) Suppose $T : \mathbb{P}_3 \to \mathbb{P}_3$ is linear and $\mathcal{B} = \{1, t, t^2\, t^3\}$.

Given that $[T]_{\mathcal{B}} = \begin{bmatrix} 2 & 0 & -1 & 1 \\ 1 & 0 & 1 & 1 \\ 2 & -3 & 0 & 2 \\ 1 & 1 & 1 & 0 \end{bmatrix}$, find $T(3 - 2t + t^3)$.

$$\begin{bmatrix} 2 & 0 & -1 & 1 \\ 1 & 0 & 1 & 1 \\ 2 & -3 & 0 & 2 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ 14 \\ 1 \end{bmatrix} = [T(3-2t+t^3)]_{\mathcal{B}}$$

$\therefore T(3-2t+t^3) = \boxed{7 + 4t + 14t^2 + t^3}$

11. (8 points)Let A be a 4×4 matrix with 4 distinct eigenvalues. One of those eigenvalues is 0.

(a) Is A diagonalizable? Why or why not.

Yes - any $n \times n$ matrix with n distinct λ-value is D-able since each corresp λ-vect is LI

→ Basis of n LI λ-vects

(b) Is A invertible? Explain your reasoning.

No 0 is a λ-val

So $\exists$ nonzero $\vec{v}$ st $A\vec{v} = 0 \cdot \vec{v} = \vec{0}$

This is a dep rel among cols of A →

cols of A not LI → A not invertible

$A \neq I$ and

12. (5 points) Let A be a non -zero square matrix such that $A^2 = A$. Show that the eigenvalues of A are either 0 or 1.

Suppose $\exists$ non-zero v with $Av = \lambda v$.

Then $A^2 = A \Rightarrow A^2 v = \lambda v$

but $A^2 v = A(Av) = A(\lambda v) = \lambda^2 v$

$\therefore$ $\lambda v = \lambda^2 v \Rightarrow \lambda = \lambda^2 \Rightarrow \lambda^2 - \lambda = 0$

$\lambda(\lambda - 1) = 0 \Rightarrow \lambda = 0$ or $\lambda = 1$ $\boxtimes$

Math 70 Final exam May 3, 2013

Name ______________________________

Instructor ______________________________

I pledge that I have neither given nor received assistance on this exam.

Signature ______________________________