Dec 13, 2007

**SOLUTIONS** (Provided by students in Math 46 - 01, Spring 2012)

1. Let us first consider the eigenvalue $\lambda_1 = 1$. To find the eigenspace, we solve the system $(A - I_3)\mathbf{x} = \mathbf{0}$.

$$(A - I_3)\mathbf{x} = \begin{bmatrix} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Note: the last column of zeros in A-I_3 doesn't belong.

The first eigenvalue is then

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

For the eigenvector $\lambda_2 = -2$, we have $(A + 2I)\mathbf{x} = \mathbf{0}$. The eigenspace is spanned by the two (linearly independent) vectors

$$\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

This three vectors would give us a basis for $\mathbb{R}^3$, and if we put them as the columns of a $3 \times 3$ matrix $P$, we would have $A = PDP^{-1}$, for $D$ a diagonal matrix.

We are asked to give an orthogonal matrix, i.e. a matrix whose columns form an orthonormal set. For that, we only need to apply the Gram-Schmidt process to the basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

First, let us construct a basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ such that $\mathbf{u}_i \perp \mathbf{u}_j$, $1 \leq i, j \leq 3, i \neq j$.

We define:

In the eigenspace for $\lambda_1$,

$$\mathbf{u}_1 = \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

In the eigenspace for $\lambda_2$,

$$\mathbf{u}_2 = \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

The component of $\mathbf{v}_3$ orthogonal to $\mathbf{v}_2$ is

$$\mathbf{u}_3 = \mathbf{v}_3 - \frac{\mathbf{v}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

The set $\{\mathbf{u}_2, \mathbf{u}_3\}$ is now orthogonal.

NOTE: in fall 2018, we did not cover orthogonal matrices, so you would be expected to just find P=[v_1 v_2 v_3] and D (the diagonal matrix with diagonal entries 1, -2,-2 and note that A=PDP^{-1} .
To find an orthogonal matrix, one needs to find an orthonormal basis of the eigenspace for lambda _1=1 and an orthonormal basis of the eigenspace for lambda_2=-2 and that's what they are doing at the bottom of the page and on the next page.

We are asked for an orthonormal basis, so we need to normalize all vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$:

$$\mathbf{w}_1 = \frac{\mathbf{u}_1}{||\mathbf{u}_1||} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$
$$\mathbf{w}_2 = \frac{\mathbf{u}_2}{||\mathbf{u}_2||} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$
$$\mathbf{w}_3 = \frac{\mathbf{u}_3}{||\mathbf{u}_3||} = \begin{bmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ \sqrt{2/3} \end{bmatrix}$$

By the diagonalization theorem, we have that $A = UDU^{-1}$, for $D$ a diagonal matrix. The matrix $U = \begin{bmatrix} \mathbf{w}_1 & \mathbf{w}_2 & \mathbf{w}_3 \end{bmatrix}$, and since it is square and orthogonal $U^{-1} = U^T$. We have then that $D = U^{-1}AU = U^TAU$.

2. The Gram-Schmidt algorithm computes an orthogonal basis for the subspace spanned by a nonorthogonal set of vectors. For that:

$$\mathbf{v}_1 = \mathbf{u}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\mathbf{v}_2 = \mathbf{u}_2 - \frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$$

$$\mathbf{v}_3 = \mathbf{u}_3 - \frac{\mathbf{u}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{u}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - 2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

3. (a) We are asked to find the projection of the vector $\mathbf{y}$ on $W = \text{span}\,\{\mathbf{u}_1, \mathbf{u}_2\}$.

$$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 = \frac{\begin{bmatrix} 0 \\ 5 \\ -4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}}\mathbf{u}_1 + \frac{\begin{bmatrix} 0 \\ 5 \\ -4 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}}{\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}}\mathbf{u}_2$$
$$= \frac{6}{6}\mathbf{u}_1 + \frac{-5}{5}\mathbf{u}_2 = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$$

(b) We want to find $\mathbf{z} \in W^{\perp}$ such that $\mathbf{y} = \mathbf{w} + \mathbf{z}$. From (a) and the orthogonal decomposition theorem, we have that $\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}} = \begin{bmatrix} 0 \\ 5 \\ -4 \end{bmatrix} - \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix}$.

(c) $||\mathbf{y} - \hat{\mathbf{y}}|| = \sqrt{30}$

(d) Vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ are clearly linearly independent, as one is not multiple of the other. Now, since $\mathbf{z}$ is in the orthogonal complement to $W$, it is orthogonal to $\mathbf{u}_1$ and $\mathbf{u}_2$. Orthogonal vectors are linearly independent. Therefore, $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{z}$ are linearly independent.

(e) $\frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} = \frac{1}{\sqrt{1^2+2^2+1^2}} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$

4. (To solve this problem you need material in the section that deals with complex eigenvalues. We have not covered that material this year.)

5. Let $f(x) = (x-\lambda_1)^{m_1} \cdots (x-\lambda_d)^{m_d}$ be the characteristic polynomial of a matrix $A$. Then $A$ is diagonalizable if for each $i = 1, 2, \ldots, d$, the dimension of the eigenspace associated to the eigenvalue $\lambda_i$ equals $m_i$.

6. (a) $[T]_{\mathcal{B}} = \begin{bmatrix} [-\mathbf{v}_1]_{\mathcal{B}} & [\mathbf{v}_1+\mathbf{v}_2]_{\mathcal{B}} \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$

(b)

$$\begin{aligned} T(x_1\mathbf{v}_1 + x_2\mathbf{v}_2) &= x_1 T(\mathbf{v}_1) + x_2 T(\mathbf{v}_2) \\ &= -x_1\mathbf{v}_1 + x_2(\mathbf{v}_1+\mathbf{v}_2) \\ &= (-x_1+x_2)\mathbf{v}_1 + x_2\mathbf{v}_2 \\ &= x_1\mathbf{v}_1 + x_2\mathbf{v}_2 \end{aligned}$$

$$\Longrightarrow \quad \begin{aligned} -x_1 + x_2 &= x_1 \\ 2x_1 &= x_2 \\ x_1 &= 1 \\ x_2 &= 2 \end{aligned}$$

(c) We are looking for vectors $\mathbf{u}_1$ and $\mathbf{u}_2$, such that $T(\mathbf{u}_1) = \lambda_1\mathbf{u}_1$ and $T(\mathbf{u}_2) = \lambda_2\mathbf{u}_2$. By hypothesis, we know that $T(\mathbf{v}_1) = -\mathbf{v}_1$. So, $\mathbf{u}_1 = \mathbf{v}_1$ is an eigenvector associated to the eigenvalue $\lambda_1 = -1$.

$$T(x_1\mathbf{v}_1 + x_2\mathbf{v}_2) = x_1\mathbf{v}_1 + x_2\mathbf{v}_2$$

holds for $x_1 = 1$, $x_2 = 2$, and therefore

$$T(\mathbf{v}_1 + 2\mathbf{v}_2) = \mathbf{v}_1 + 2\mathbf{v}_2.$$

This means that $\mathbf{u}_2 = \mathbf{v}_1 + 2\mathbf{v}_2$ is an eigenvector for the eigenvalue $\lambda_2 = 1$. The basis is therefore

$$\{\mathbf{u}_1, \mathbf{u}_2\} = \{\mathbf{v}_1, \mathbf{v}_1 + 2\mathbf{v}_2\}.$$

7. (a) The matrix $P_{S\leftarrow\mathcal{B}}$ is the matrix whose column vectors are the vectors in the basis $\mathcal{B}$ expressed in terms of the standard basis $S$. That is, the matrix

$$\begin{bmatrix} 1 & 5 \\ 1 & 4 \end{bmatrix}.$$

(b) The matrix $P_{\mathcal{B}\leftarrow\mathcal{S}}$ is just the inverse of the matrix in part (a). For that we can either use the formula for the inverse of a $2\times 2$ matrix, or just the algorithm:

$$\begin{bmatrix} 1 & 5 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 1 & 0 \\ 0 & -1 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 1 & 0 \\ 0 & 1 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -4 & 5 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

From here we have that

$$P_{\mathcal{B}\leftarrow\mathcal{S}} = P_{\mathcal{S}\leftarrow\mathcal{B}}^{-1} = \begin{bmatrix} -4 & 5 \\ 1 & -1 \end{bmatrix}$$

(c) $T(\mathbf{v}_1) = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $T(\mathbf{v}_2) = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \end{bmatrix}$.

(d) $[T(\mathbf{v}_1)]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}_{\mathcal{B}}$ and $[T(\mathbf{v}_2)]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}_{\mathcal{B}}$.

(e) $[T]_{\mathcal{B}} = [[T(\mathbf{v}_1)]_{\mathcal{B}} \quad [T(\mathbf{v}_2)]_{\mathcal{B}}] = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}$.

(f) $[T]_{\mathcal{B}}[\mathbf{w}]_{\mathcal{B}} = [T(\mathbf{w})]_{\mathcal{B}}$ for all $\mathbf{w} \in \mathbb{R}^2$.

(g) Let us denote by $\mathbb{R}^2_{\mathcal{B}}$, respectively $\mathbb{R}^2_{\mathcal{S}}$, the vector space $\mathbb{R}^2$ with basis $\mathcal{B}$, respectively $\mathcal{S}$. If we look at the matrices as standard matrices of linear transformations, we have the diagram:

$$\begin{tikzcd} \mathbb{R}^2_{\mathcal{B}} \arrow[r, "{[T]_{\mathcal{B}}}"] \arrow[d, "P"'] & \mathbb{R}^2_{\mathcal{B}} \\ \mathbb{R}^2_{\mathcal{S}} \arrow[r, "A"'] & \mathbb{R}^2_{\mathcal{S}} \arrow[u, "P^{-1}"'] \end{tikzcd}$$

From here it is clear that $[T]_{\mathcal{B}} = P^{-1}AP$. It is, in fact, easy to check, as we have computed all matrices explicitly:

$$\begin{bmatrix} -4 & 5 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 5 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}.$$

8. (a) $A$ is similar to $B$ if there exists an invertible matrix $P$, such that $P^{-1}AP = B$.
   (b) An eigenvector of $A$ is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$, for some scalar $\lambda$.
   (c) The transformation $T : V \to W$ is one-to-one if each $\mathbf{b} \in W$ is the image of at most one $\mathbf{x} \in V$.
   (d) An indexed set of vectors, $\mathbf{v}_1, \ldots, \mathbf{v}_p$ is linearly independent if the vector equation $x_1\mathbf{v}_1 + \cdots + x_p\mathbf{v}_p$ has only the trivial solution.

9. Any of the properties listed in the invertible matrix theorem is equivalent to the fact that $\det A \neq 0$, for $A$ an $n \times n$ matrix. In particular, the matrix is invertible, $A$ is row equivalent to the $n \times n$ identity matrix, and the columns of $A$ are linearly independent.

10. (a) Associative law of scalar multiplication.
    (b) The transpose of a product: $(AB)^T = B^TA^T$
    (c) $A^T = A$ because $A$ is a symmetric matrix.
    (d) Associative law of vector-matrix multiplication
    (e) $v_1 \cdot v_2 = 0$ and $v_1$ and $v_2$ are orthogonal.