Math 46
Solutions for Final Exam Review

1. In the interests of brevity, I'm not going to write out either the question or the answer to the various parts of problem one. The answers to the parts (a), (b), and (c) should all be in your notes, and as for part (d), well ... I don't think I should answer that just yet. Maybe after the exam.

2. Let $A \in M_{m\times n}$. Assume that for all $\mathbf{x} \in \mathbf{R}^n$ that $A\mathbf{x} = \mathbf{0}$. Prove that A is the zero matrix.

Solution: Well, we know that $A\mathbf{x} = \mathbf{0}$ for every single vector $\mathbf{x} \in \mathbf{R}^n$. That means we can pick any $\mathbf{x}$ we want, and plug it in. Me, I want to pick $\mathbf{x} = \mathbf{e}_i$. Because then we see that $A\mathbf{x} = A\mathbf{e}_i$ is the ith column of A. But that's equal to $\mathbf{0}$ by hypothesis! So every column of A is zero, so the whole matrix has to be zero. ♣

3. Solve the following linear system:

$$\begin{array}{rcrcrcr} 2x & + & y & - & 2z & = & 10 \\ 3x & + & 2y & + & 2z & = & 1 \\ 5x & + & 4y & + & 3z & = & 4 \end{array}$$

(a) by row reduction.

(b) by Cramer's Rule.

Solution: I'm going to omit the details of this, since you can see multitudinous examples of row reduction and Cramer's Rule in the textbook and in your notes. However, the answer is that there is only one solution to this system, and it's given by $x = 1$, $y = 2$, and $z = -3$. ♣

4. Let A be an $m\times n$ matrix and let $T_A\colon \mathbf{R}^n \to \mathbf{R}^m$ be defined by $T_A(\mathbf{x}) = A\mathbf{x}$. Are the following statements true or false? If true, give a proof. If false, explain why.

(a) $\dim \mathrm{Nul} A \le n$.

Solution: True. The null space of A is a subspace of $\mathbf{R}^n$, so its dimension has to be at most n. ♣

(b) $\text{rank}A \leq m$.

Solution: True. The rank of A is equal to the number of pivots in the reduced row-echelon form of A. Since each pivot is in a different row, the number of pivots is no more than the number of rows of A, which is m. ♣

(c) If $n > m$ then the linear transformation T_A cannot be one-to-one.

Solution: True. Recall that T_A is one-to-one if and only if A has a pivot in every column (after row-reduction). This is only possible if A has at least as many rows as columns ... which means that if T_A is one-to-one, then we can't have $n > m$. ♣

(d) If $n < m$ then T_A cannot be onto.

Solution: True. Recall that T_A is onto if and only if A has a pivot in every row (after row-reduction). This is only possible if A has at least as many columns as rows ... which means that if T_A is onto, then we can't possibly have $n < m$. ♣

(e) If T_A is one-to-one and $m = n$, then T_A must be onto.

Solution: True. (Yes, all the answers to question four were "true".) If $m = n$, then A is a square matrix. If A is square, then the IMT applies, and that means that T_A is onto if and only if it's one-to-one. So we're done. ♣

5. Let A be an $n \times n$ matrix satisfying $A^3 = I_n$, where I_n is the $n \times n$ identity matrix. Show that $\det A = 1$.

Solution: Since $A^3 = I$, we know that $\det A^3 = \det I = 1$. This means that $(\det A)^3 = 1$, since $\det(AB) = \det(A)\det(B)$ for any two matrices A and B. But if $(\det A)^3 = 1$, then assuming that the entries in A are real numbers (which, in this course, they always will be), we must have $\det A = 1$, as desired. ♣

6. For the following problems, prove the statement or give a specific counterexample:

(a) $W = \{p \in P_2 \mid (p(3))^2 + p(3) = 0\}$ is a subspace of P_3.

Solution: This is obviously not true, since W is defined to be a set of polynomials in P_2, so it can't be a subspace of P_3.

If you ignore that typo, and ask if W is a subspace of P_2 instead, then the answer is still no. To check if W is a subspace, you need to check three things: the existence of $\mathbf{0} \in W$, closure under addition, and closure under scalar multiplication. As it turns out, the zero vector is in W, but W fails the second and third tests.

For example, let $p(t) = -1$, the constant polynomial. Then certainly $(p(3))^2 + p(3) = (-1)^2 + (-1) = 1 - 1 = 0$, so $p(t) \in W$. But if we multiply $p(t)$ by the scalar 3, we see that $(3p(3))^2 + 3p(3) = 9 + 3(-1) = 6 \neq 0$. Thus, $3p(t)$ is not in W, so W is not closed under scalar multiplication. ♣

(b) $W = \{(x, y) \in \mathbf{R}_2 \mid x + y = 0\}$ is a subspace of $\mathbf{R}_2$.

Solution: This is true. Recall that to check if W is a subspace of $\mathbf{R}_2$, we need to check three things: the existence of $\mathbf{0} \in W$, closure under addition, and closure under scalar multiplication. Let's check them, one by one.

First, it is clear that $\mathbf{0} = (0, 0)$ is in W. This follows immediately from the fact that $0 + 0 = 0$.

Second, we need to check that W is closed under addition. To do this, pick any two vectors (x_1, y_1) and (x_2, y_2) in W. We want to show that $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ is also in W. This means showing that $x_1 + x_2 + y_1 + y_2 = 0$, because that's the definition of W.

But we already know that $x_1 + y_1 = 0$, because $(x_1, y_1) \in W$. And $x_2 + y_2 = 0$, because $(x_2, y_2) \in W$. So that means that $x_1 + x_2 + y_1 + y_2 = 0$, so $(x_1 + x_2, y_1 + y_2) \in W$, as desired.

Lastly, we need to check that W is cloesd under scalar multiplication. To do this, pick any vector $(x, y) \in W$, and pick any scalar $\alpha \in \mathbf{R}$. We want to show that $\alpha(x, y) = (\alpha x, \alpha y)$ is in W.

This amounts to showing that $\alpha x + \alpha y = 0$. But that follows immediately from the fact that $x + y = 0$ (which is the definition of what it means for (x, y) to be in W), so we're done.

Since we've shown that W satisfies all three criteria for being a subspace of $\mathbf{R}_2$, we're done. ♣

(c) Let V be a vector space and let $f_1 \colon V \to \mathbf{R}$ and $f_2 \colon V \to \mathbf{R}$ be linear transformations. Define $T \colon V \to \mathbf{R}_2$ by $T(\mathbf{v}) = (f_1(\mathbf{v}), f_2(\mathbf{v}))$. Then T is linear.

Solution: This is true.

In order to prove that T is linear, we need to check two things: that T respects addition, and that it respects scalar multiplication.

First, we need to show that T respects addition. This means showing that $T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$ for any pair of vectors $\mathbf{x}$ and $\mathbf{y}$ in V. This is done as follows:

$$\begin{aligned} T(\mathbf{x}+\mathbf{y}) &= (f_1(\mathbf{x}+\mathbf{y}), f_2(\mathbf{x}+\mathbf{y})) \\ &= (f_1(\mathbf{x}) + f_1(\mathbf{y}), f_2(\mathbf{x}) + f_2(\mathbf{y})) \\ &= (f_1(\mathbf{x}), f_2(\mathbf{x})) + (f_1(\mathbf{y}), f_2(\mathbf{y})) \\ &= T(\mathbf{x}) + T(\mathbf{y}) \end{aligned}$$

(Remember that f_1 and f_2 are both linear – that's the trick to going from the first to the second line.)

All that's left to do is to check that T respects scalar multiplication. This means checking that $T(\alpha\mathbf{x}) = \alpha T(\mathbf{x})$ for any vector $\mathbf{x} \in V$ and any scalar α. To wit:

$$\begin{aligned} T(\alpha\mathbf{x}) &= (f_1(\alpha\mathbf{x}), f_2(\alpha\mathbf{x})) \\ &= (\alpha f_1(\mathbf{x}), \alpha f_2(\mathbf{x})) \\ &= \alpha(f_1(\mathbf{x}), f_2(\mathbf{x})) \\ &= \alpha T(\mathbf{x}) \end{aligned}$$

so we're done! ♣

7. Let $S = \left\{ \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 8 & 2 \\ 4 & 6 \end{bmatrix} \right\}$.

(a) Decide whether S is independent.

Solution: No, S is not independent, because we have the linear relation:

$$2\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 2 \\ 4 & 6 \end{bmatrix}$$

Now, if you didn't magically spot this, you probably needed to use a coordinate mapping to $\mathbf{R}^4$. Using the following basis for $M_{2\times2}$:

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

the three vectors in S correspond to the following vectors in $\mathbf{R}^4$:

$$[S]_{\mathcal{B}} = \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 8 \\ 2 \\ 4 \\ 6 \end{pmatrix} \right\}$$

The independence of S is equivalent to the independence of $[S]_{\mathcal{B}}$; that is, S is linearly independent if and only if $[S]_{\mathcal{B}}$ is linearly independent.

But we have an easy way of checking the independence of vectors in $\mathbf{R}^4$, which is to make a matrix A whose columns are the vectors in $[S]_{\mathcal{B}}$ and row reduce. $[S]_{\mathcal{B}}$ is linearly independent if and only if the reduced row-echelon form of A has a pivot in every column. In our case, the third column is missing a pivot (I'm omitting all the calculations here), so it's dependent on the first two columns, so $[S]_{\mathcal{B}}$ is linearly dependent, so S is also linearly dependent. ♣

(b) Let $W = \text{Span } S$. Use the result of (a) to find a basis of W that is a subset of S. Find $\dim W$.

Solution: From the solution to part (a), we know that we can find such a basis by just deleting the vectors in S that correspond to non-pivot columns in A. This means deleting just the last vector, so the desired basis is:

$$\mathcal{B} = \left\{ \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \right\}$$

Thus, the dimension of W is 2, the number of vectors in a basis of W. ♣

(c) Determine whether $\begin{bmatrix} 5 & 1 \\ -1 & 9 \end{bmatrix} \in W = \text{Span } S$.

Solution: Once again, we use the coordinate mapping to $\mathbf{R}^4$. Let's give this vector a name – call it $\mathbf{v}$. Using the same basis $\mathcal{B}$ as before, we convert $\mathbf{v}$ into the following column vector:

$$[\mathbf{v}]_{\mathcal{B}} = \begin{pmatrix} 5 \\ 1 \\ -1 \\ 9 \end{pmatrix}$$

By the properties of the coordinate mapping, we know that $\mathbf{v}$ is in the span of S if and only if $[\mathbf{v}]_{\mathcal{B}}$ is in the span of $[S]_{\mathcal{B}}$. This last question is easy to resolve: just make an augmented matrix A whose first two columns are in the basis of W we found above, and whose last column is $[\mathbf{v}]_{\mathcal{B}}$. Put A in row-echelon form. If there is a pivot in the last column, then $[\mathbf{v}]_{\mathcal{B}}$ will *not* be in the span of $[S]_{\mathcal{B}}$, and if there is no pivot in the last column, then $[\mathbf{v}]_{\mathcal{B}}$ *will* be in the span of $[S]_{\mathcal{B}}$.

As it happens (and I'm again omitting all the calculations), you do find a pivot in the last column, so $[\mathbf{v}]_{\mathcal{B}}$ is not in the span of $[S]_{\mathcal{B}}$, and therefore $\mathbf{v}$ is not in the span of S. ♣

8. If you are given a square upper triangular matrix, how would you tell at a glance whether or not it is invertible? Explain your answer using determinants. How would you tell at a glance the eigenvalues of the matrix and their multiplicities?

Solution: The determinant of a triangular matrix is the product of the diagonal entries. By the IMT, a matrix is invertible if and only if its determinant is not zero. Therefore, an upper triangular matrix is invertible if and only if all of its diagonal entries are nonzero.

Moreover, the eigenvalues of a triangular matrix are precisely its diagonal entries, and the number of times each eigenvalue appears is precisely its algebraic multiplicity. Therefore, you can read the eigenvalues of an upper triangular matrix and their multiplicities directly off the diagonal. ♣

9. Let V and W be vector spaces and let $T: V \to W$ be a linear transformation.

(a) Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_\ell\}$ be a set of vectors that spans V. Prove that the set $T(S) = \{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3), \ldots, T(\mathbf{v}_\ell)\}$ spans range T in W.

Solution: Let $\mathbf{x}$ be any vector in range T. Then by the definition of range T, we know that $\mathbf{x}$ can be written in the form $\mathbf{x} = T(\mathbf{v})$ for some $\mathbf{v} \in V$.

Now, S spans V, so we can write $\mathbf{v} = c_1\mathbf{v}_1 + \ldots + c_\ell\mathbf{v}_\ell$ as a linear combination of the elements of S. Applying T to both sides of this expression gives $\mathbf{x} = T(\mathbf{v}) = T(c_1\mathbf{v}_1 + \ldots + c_\ell\mathbf{v}_\ell) = c_1T(\mathbf{v}_1) + \ldots + c_\ell T(\mathbf{v}_\ell)$, so $\mathbf{x}$ can be written as a linear combination of the set $T(S)$. This means that $\mathbf{x}$ is in the span of $T(S)$, so $T(S)$ spans range T, as desired. ♣

(b) Assume T is one-to-one, and assume S is a basis of V. Prove that the set $T(S)$ is independent.

Solution: First, let's do the straightforward solution. Consider the equation $c_1T(\mathbf{v}_1) + \ldots c_\ell T(\mathbf{v}_\ell) = \mathbf{0}$, and let's try to prove that $c_1 = \ldots = c_\ell = 0$. If we can do that, then we'll have shown that $T(S)$ is linearly independent, because we'll have shown that if a linear combination of elements of $T(S)$ equals zero, then that linear combination has all its coefficients equal to zero.

So, assume $c_1T(\mathbf{v}_1) + \ldots c_\ell T(\mathbf{v}_\ell) = \mathbf{0}$. Then because T is linear, we have $T(c_1\mathbf{v}_1 + \ldots + c_\ell\mathbf{v}_\ell) = \mathbf{0}$. But T is one-to-one, so we can conclude that $c_1\mathbf{v}_1 + \ldots + c_\ell\mathbf{v}_\ell = \mathbf{0}$. But we know that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_\ell\}$ is a basis of V, so in particular it's linearly independent! Which means that $c_1 = \ldots = c_\ell = 0$, which is what we wanted. Thus, $T(S)$ is linearly independent, as desired.

Now, that's not how the professionals would prove this. They'd argue as follows. Since T is one-to-one, its nullity must be zero, so by Rank-Nullity $\mathrm{rank} T$ must equal $\dim V = \ell$. By part (a), we know that $T(S)$ spans range T. We also know that it contains ℓ vectors. Therefore, it must be a basis of range T, and therefore in particular must be linearly independent. ♣

(c) Define $L\colon P_3 \to \mathbf{R}_3$ by $L(p) = (p(0), p(1), p(3))$. Show that L is a linear transformation.

Solution: We have two things to check:

$$\begin{aligned} L(p+q) &= ((p+q)(0), (p+q)(1), (p+1)(3)) \\ &= (p(0)+q(0), p(1)+q(1), p(3)+q(3)) \\ &= (p(0), p(1), p(3)) + (q(0), q(1), q(3)) \\ &= L(p) + L(q) \end{aligned}$$

and

$$\begin{aligned} L(\alpha p) &= ((\alpha p)(0), (\alpha p)(1), (\alpha p)(3)) \\ &= (\alpha p(0), \alpha p(1), \alpha p(3)) \\ &= \alpha(p(0), p(1), p(3)) \\ &= \alpha L(p) \end{aligned}$$

Since both criteria are satisfied by L, it must be linear, by definition. ♣

(d) Use the result of (a) and the basis $\mathcal{B} = \{1, t, t^2, t^3\}$ of P_3 to find a spanning set of range L. Find a subset of this spanning set that is a basis of range L.

Solution: By part (a), we know that $L(\mathcal{B})$ is a spanning set of range L. We calculate thus:

$$L(\mathcal{B}) = \{L(1), L(t), L(t^2), L(t^3)\} = \{(1,1,1), (0,1,3), (0,1,9), (0,1,27)\}$$

By using a coordinate mapping (or by another method, explained below), you find that the first three vectors in $T(\mathcal{B})$ are linearly independent, and that the fourth depends on the first three ($(0,1,27) = (0,1,3)+4((0,1,9)-(0,1,3))$). Thus, the first three vectors in $T(\mathcal{B})$ are a basis of range L.

Alternatively, you could reason as follows. Let's say that $p(t) \in P_3$ was a polynomial in the kernel of L. Then we'd have $L(p) = \mathbf{0}$, which means $(p(0), p(1), p(3)) = (0,0,0)$. This means that 0,1, and 3 are all roots of $p(t)$! So we can factor $p(t) = c(t)(t-1)(t-3) = c(t^3 - 4t^2 + 3t)$ for some constant c.

This means we know exactly what the kernel of L is – it's precisely the multiples of the polynomial $p(t) = t^3 - 4t^2 + 3t$! In particular, it's one-dimensional, so the nullity of L is 1. By Rank-Nullity, this means that the rank of L is 3. Since the last vector in $L(\mathcal{B})$ is dependent on the first three (see the observation above), we know that the first three vectors in $L(\mathcal{B})$ span range L. But range L is three-dimensional, so they must be a basis of range L. ♣

10. (a) Let $T \colon V \to W$ be a linear transformation. State the Rank plus Nullity Theorem for T.

Solution: It's rank T + nullity $T = \dim V$. ♣

(b) Let $T \colon M_{2\times 2} \to \mathbf{R}^6$ be a linear transformation. If nullity $T \leq 2$, what are the possible values of rank T?

Solution: We know that the dimension of $M_{2\times 2}$ is 4. Therefore, by the Rank-Nullity Theorem stated above, it follows that if the nullity of T is at most 2, then the rank of T must be at least 2, but no bigger than 4 (since rank $T \leq \dim M_{2\times 2}$). ♣

(c) Let V be a finite dimensional vector space and let $T \colon V \to \mathbf{R}^5$ be a linear transformation. If T is onto and nullity T is 3, what is $\dim V$?

Solution: Since T is onto, we know that rank $T = \dim \mathbf{R}^5 = 5$. We're told that the nullity of T is 3. Therefore, by Rank-Nullity, $\dim V$ must be equal to 5+3=8. ♣

(d) Let V be a finite dimensional vector space and let $T \colon V \to V$ be linear. Prove that if T is one-to-one then T is onto.

Solution: Let $\mathcal{B}$ be a basis of V. Then T is one-to-one if and only if the matrix $[T]_{\mathcal{B}}$ has a pivot in every column (after row reduction). The matrix $[T]_{\mathcal{B}}$ is square, since T goes from V to itself. Therefore we can conclude that $[T]_{\mathcal{B}}$ has a pivot in every row. But this implies immediately that T has to be onto. ♣

11. Let $A = \begin{bmatrix} -5 & -9 & -6 \\ 0 & -2 & 0 \\ 3 & 9 & 4 \end{bmatrix}$.

(a) Decide whether A is diagonalizable. If so, find a diagonal matrix D and an invertible matrix P such that $D = P^{-1}AP$.

Solution: First, we need to find the characteristic polynomial of A, which is $\det(A - \lambda I) = -\lambda^3 + 5\lambda^2 - 8\lambda + 4 = -(\lambda - 1)(\lambda + 2)^2$. (If you had trouble factoring that polynomial, bear in mind that all the roots of a polynomial have to divide evenly into the constant coefficient. So when you're looking for roots, try plugging in plus or minus the factors of the constant coefficient. In this case, that means the factors of 4, which are ± 1, ± 2, and ± 4.)

Anyway, the roots of the polynomial are 1 and -2, so those are the eigenvalues. Now we need to find bases for the corresponding eigenspaces. Always check the eigenvalues in descending order of multiplicity when you're doing a diagonalization, because if the matrix is not diagonalizable, it'll be the eigenvalues of high multiplicity that will let you know.

So, let's start by finding a basis for the (-2)-eigenspace. This means finding a basis for the nullspace of the matrix $A + 2I$:

$$A + 2I = \begin{bmatrix} -3 & -9 & -6 \\ 0 & 0 & 0 \\ 3 & 9 & 6 \end{bmatrix}$$

It's pretty easy to see from here that the nullspace of $A+2I$ is two-dimensional, and that a basis for said nullspace is given by:

$$\left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}$$

At this point, we know that the matrix is going to turn out diagonlizable, because the (-2)-eigenspace is 2-dimensional, and we know we're going to get at least a one-dimensional eigenspace out of $\lambda = 1$ (eigenspaces are never

0-dimensional), which makes three in total, which is all we need. Which is a pity, because if A were not diagonalizable, we'd have nothing to do in part (b). But which, for the same reason, makes it not surprising that A is diagonalizable.

Anyway, back to mathematics. We still have to find the 1-eigenspace of A. This is the same as the nullspace of $A - I$:

$$A - I = \begin{bmatrix} -6 & -9 & -6 \\ 0 & -3 & 0 \\ 3 & 9 & 3 \end{bmatrix}$$

After a bit of calculation, you find that this nullspace is one-dimensional, and a basis for it is given by:

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$$

The diagonal matrix D and the invertible matrix P that we want are simply the eigenvalues and eigenvectors of A, respectively, arranged appropriately and in the same order:

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \qquad P = \begin{bmatrix} 1 & -3 & -2 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

(b) If A is diagonalizable, use the result of (a) to find A^{10}.

Solution: We know that $D = P^{-1}AP$, so that $A = PDP^{-1}$, so that $A^{10} = PD^{10}P^{-1}$. We can easily calculate D^{10}:

$$D^{10} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1024 & 0 \\ 0 & 0 & 1024 \end{bmatrix}$$

Thus, we get $A^{10} = PD^{10}P^{-1} = \begin{bmatrix} 2047 & 3069 & 2046 \\ 0 & 1024 & 0 \\ -1023 & -3069 & -1022 \end{bmatrix}$. ♣

12. Determine whether each of the following matrices is diagonalizable. Justify your answer. If A is diagonalizable, find a diagonal matrix similar to A.

(a) $A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Solution: No, it is not diagonalizable. The eigenvalues of A are -1, 3, and 1, and only 3 has a multiplicity higher than 1, so we check that eigenspace first. If its dimension is any smaller than 2 (the algebraic multiplicity), then we know that A is not diagonalizable.

So, we compute the null space of $A - 3I$, and check its dimension. We get:

$$A - 3I = \begin{bmatrix} -4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

This matrix plainly has three pivots, so it has rank three, so its nullity is but one. This means that the 3-eigenspace of A is only one-dimensional, so since 3 has algebraic multiplicity 2, it follows that A is not diagonalizable. ♣

(b) $A = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Solution: Yes, this matrix is diagonalizable. Again, the eigenvalues are -1, 3, and 1, and again only 3 has multiplicity higher than one. Therefore, as long as the 3-eigenspace is 2-dimensional, we'll know that A is diagonalizable.

So, we check the dimension of the null space of $A - 3I$:

$$A - 3I = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This matrix has only two pivots, so its rank is two, so its nullity must also be two. Therefore, the 3-eigenspace in this case is 2-dimensional, so since the algebraic multiplicity of 3 is two, this means that A is diagonalizable.

Sadly, this means we have extra work to do, namely, to find a diagonal matrix similar to A. But this is easy – just list the eigenvalues of A down the diagonal:

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

13. Let A be a 4×4 matrix with characteristic polynomial $(\lambda^2 - 1)(\lambda - 3)^2$. Then A is diagonalizable if and only if the 3-eigenspace has what dimension?

Solution: This is the theory behind problem 12. The answer is two, because we need the algebraic multiplicity of each eigenvalue to equal the dimension of the corresponding eigenspace. For eigenvalues of multiplicity one, this is automatically true, so all we need to worry about is the eigenvalue 3, which has multiplicity two. ♣

14. Let A be a matrix with eigenvalues 1, -2, and 4. Is there necessarily a matrix B such that $B^2 = A$? Prove your answer.

Solution: No, there cannot be such a matrix B, as long as B has real entries. The easiest way to see this is to notice that the determinant of A is the product of its eigenvalues, which is -8. If there were a matrix B such that $B^2 = A$, then we'd have $-8 = \det A = \det B^2 = (\det B)^2$, which is plainly impossible if $\det B$ is a real number! So there can be no such B. ♣

15. Define a linear transformation $T\colon P_2 \to P_2$ by

$$T(a + bt + ct^2) = (a - b + 3c) + (2b + c)t + 3ct^2$$

for all $a + bt + ct^2 \in P_2$ and let $\mathcal{B} = \{1, t, t^2\}$ be the standard basis of P_2.

(a) Find $A = [T]_{\mathcal{B}}$.

Solution: We can calculate this using the standard technique:

$$\begin{aligned} [T]_{\mathcal{B}} &= \left[\ [T(1)]_{\mathcal{B}} \mid [T(t)]_{\mathcal{B}} \mid [T(t^2)]_{\mathcal{B}}\ \right] \\ &= \left[\ [1]_{\mathcal{B}} \mid [-1 + 2t]_{\mathcal{B}} \mid [3 + t + 3t^2]_{\mathcal{B}}\ \right] \\ &= \begin{bmatrix} 1 & -1 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix} \end{aligned}$$

(b) Determine whether T is diagonalizable. If yes, find a basis $\mathcal{C}$ of P_2 such that the matrix of T with respect to $\mathcal{C}$, namely $[T]_{\mathcal{C}}$, is a diagonal matrix, and find $[T]_{\mathcal{C}}$. If T is not diagonalizable, explain why not.

Solution: We know that T is diagonalizable if and only if $[T]_{\mathcal{B}}$ is diagonalizable, so let's check if $[T]_{\mathcal{B}}$ is diagonalizable. Happily, $[T]_{\mathcal{B}}$ is an upper triangular matrix, so we can read the eigenvalues off the main diagonal: they're 1, 2, and 3. In particular, they're all different! So $[T]_{\mathcal{B}}$ has to be diagonalizable.

So let's find the eigenvectors. (We're not doing this just for fun – we're going to need to know them later.) First, let's find a basis for the 1-eigenspace. This means finding a basis for the nullspace of $A - I$:

$$A - I = \begin{bmatrix} 0 & -1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

The last two columns are both pivot columns, and the first column is zero, so the nullspace of this matrix is spanned by:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Next, let's do the 2-eigenspace. This means finding the nullspace of $A-2I$:

$$A - 2I = \begin{bmatrix} -1 & -1 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

A bit of calculation shows that the nullspace of $A - 2I$ (and hence the 2-eigenspace of A) is spanned by:

$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

Finally, there's the 3-eigenspace. We have:

$$A - 3I = \begin{bmatrix} -2 & -1 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

After a bit of calculation, you find that the nullspace of $A - 3I$ is spanned by:

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

All three of these vectors are eigenvectors of $A = [T]_{\mathcal{B}}$, with different eigenvalues, so they must be linearly independent. So, in summary, we've found the following basis of $\mathbf{R}^3$ which consists entirely of eigenvectors of $[T]_{\mathcal{B}}$:

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

At this point, all we need to do is convert these column vectors back into polynomials via the $\mathcal{B}$-coordinate mapping, and we'll have our basis $\mathcal{C}$. Thus, we have:

$$\mathcal{C} = \{1, 1 - t, 1 + t + t^2\}$$

and

$$[T]_{\mathcal{C}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

because any diagonal matrix representing T must have the eigenvalues of T down the main diagonal, in the same order as in the corresponding basis $\mathcal{C}$.

Let's explain why this works. We want a basis $\mathcal{C}$ such that $[T]_{\mathcal{C}}$ is a diagonal matrix. What we're actually claiming is the following:

Theorem 1 *Let V be a finite dimensional vector space, and let $T: V \to V$ be a linear transformation. Let $\mathcal{C}$ be a basis of V. Then $[T]_{\mathcal{C}}$ is a diagonal matrix if and only if every vector in $\mathcal{C}$ is an eigenvector of T.*

Note that in light of this theorem, once we've found a basis of V consisting entirely of eigenvectors of T, we can use that basis for $\mathcal{C}$.

Proof of Theorem: Let's say we have a basis $\mathcal{C} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of V. Then:

$$[T]_{\mathcal{C}} = \left[\; [T(\mathbf{v}_1)]_{\mathcal{C}} \mid \ldots \mid [T(\mathbf{v}_n)]_{\mathcal{C}} \; \right]$$

If all the vectors $\mathbf{v}_i$ are eigenvectors of T, then for each i, we have $T(\mathbf{v}_i) = \lambda_i \mathbf{v}_i$, where λ_i is the eigenvalue corresponding to $\mathbf{v}_i$. Thus, we have $[T(\mathbf{v}_i)]_{\mathcal{C}} =$

$\lambda_i[\mathbf{v}_i]_{\mathcal{C}} = \lambda_i\mathbf{e}_i$, since the vectors $\mathbf{v}_i$ correspond to the standard basis vectors in $\mathbf{R}^n$ via the $\mathcal{C}$-coordinate mapping. But that means that the columns of $[T]_{\mathcal{C}}$ are just multiples of the standard basis vectors, which means that $[T]_{\mathcal{C}}$ must be a diagonal matrix, with the eigenvalues down the diagonal.

Conversely, say that $[T]_{\mathcal{C}}$ is a diagonal matrix, with diagonal entries $\lambda_1, \ldots, \lambda_n$. Then the ith column of $[T]_{\mathcal{C}}$ is equal to $\lambda_i\mathbf{e}_i$, so we have $[T(\mathbf{v}_i)]_{\mathcal{C}} = \lambda_i\mathbf{e}_i = \lambda_i[v_i]_{\mathcal{C}} = [\lambda_i\mathbf{v}_i]_{\mathcal{C}}$, so $T(\mathbf{v}_i) = \lambda_i\mathbf{v}_i$, which means precisely that $\mathbf{v}_i$ is a λ_i-eigenvector of T. ♣

16. Let $W = \operatorname{span}\left\{\begin{pmatrix}1\\1\\2\end{pmatrix}, \begin{pmatrix}2\\0\\1\end{pmatrix}\right\}$.

(a) Use the Gram-Schmidt process to find an orthonormal basis of W.

Solution: The first thing to notice is that we want an orthonormal basis, not just an orthogonal one. That means that we'll have to add an extra stage to the Gram-Schmidt process described in the book – we'll need to normalise the vectors in our orthogonal basis so that they have length one. (Remember that an orthonormal basis is an orthogonal basis in which each vector has length one.)

Anyway, so first let's use the Gram-Schmidt process to find an orthogonal basis of W, and then we'll normalise. We have:

$$\begin{aligned}
\mathbf{v}_1 &= \begin{pmatrix}1\\1\\2\end{pmatrix}\\
\mathbf{v}_2 &= \begin{pmatrix}2\\0\\1\end{pmatrix} - \left[\begin{pmatrix}2\\0\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\2\end{pmatrix} \div \begin{pmatrix}1\\1\\2\end{pmatrix}\cdot\begin{pmatrix}1\\1\\2\end{pmatrix}\right]\begin{pmatrix}1\\1\\2\end{pmatrix}\\
&= \begin{pmatrix}2\\0\\1\end{pmatrix} - (4 \div 6)\begin{pmatrix}1\\1\\2\end{pmatrix}\\
&= \begin{pmatrix}4/3\\-2/3\\-1/3\end{pmatrix}
\end{aligned}$$

This gives us an orthogonal basis of W. Now, all we need to do is divide each of these vectors by their lengths, and we'll have an orthonormal basis. We have $||\mathbf{v}_1|| = \sqrt{6}$ and $||\mathbf{v}_2|| = \sqrt{21}/3$, so we get the following orthonormal basis of W:

$$\left\{ \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \frac{1}{\sqrt{21}} \begin{pmatrix} 4 \\ -2 \\ -1 \end{pmatrix} \right\}$$

(b) Find a basis of $W^\perp$.

Solution: If you have taken Math 13, you'll know that a basis for $W^\perp$ can be found simply by taking the cross product of the two given vectors. But let's solve this a different way instead.

Choose any vector in $W^\perp$:

$$\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

We know that x, y, and z satisfy the two following equations, which encode the fact that $\mathbf{v}$ is perpendicular to the two vectors which span W:

$$\begin{aligned} x + y + 2z &= 0 \\ 2x + z &= 0 \end{aligned}$$

If we solve this system of equations, we'll have found our basis for $W^\perp$. A bit of calculation shows that a basis for the solution space of this system is given by:

$$\left\{ \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} \right\}$$

So this must also be a basis for $W^\perp$. ♣

17. Let $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}$. Show that $\mathcal{B}$ is an orthogonal set.

Solution: This is easy. All we need to do is check that the two vectors are orthogonal to one another, which is a simple dot product calculation: $1 + 0 - 1 = 0$. Since the answer is 0, the two vectors are orthogonal. ♣

Let $\mathbf{x} \in \mathbf{R}^3$. Find the orthogonal projection of $\mathbf{x}$ onto $W = \text{span}\mathcal{B}$.

Solution: Write $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Denote the two vectors in $\mathcal{B}$ by $\mathbf{v}_1$ and $\mathbf{v}_2$, respectively. Then:

$$\begin{aligned}
\mathrm{proj}_W \mathbf{x} &= \frac{\mathbf{x}\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\mathbf{v}_1 + \frac{\mathbf{x}\cdot\mathbf{v}_2}{\mathbf{v}_2\cdot\mathbf{v}_2}\mathbf{v}_2 \\
&= \frac{x-z}{2}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \frac{x+2y+z}{6}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \\
&= \frac{1}{3}\begin{pmatrix} 2x+y-z \\ x+2y+z \\ -x+y+2z \end{pmatrix}
\end{aligned}$$

18. Let $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ be an orthonormal set in $\mathbf{R}^n$. Prove that S is independent.

Solution: This is straight from the book, since any orthonormal set is also an orthogonal set. See Theorem 4 on page 379 of the textbook. ♣

(a) Let $W = \mathrm{span} S$ and let $\mathbf{w} \in W$. Prove that $\mathbf{w} = (\mathbf{w}\cdot\mathbf{u}_1)\mathbf{u}_1 + \ldots + (\mathbf{w}\cdot\mathbf{u}_k)\mathbf{u}_k$.

Solution: This is also straight from the textbook, from Theorem 5 on page 379. The only thing to notice is that since S is an orthonormal set, it follows that $\mathbf{u}_i\cdot\mathbf{u}_i = 1$ for each i. ♣

(b) Why does the result of (a) show that if $\mathbf{w} \in \mathrm{span} S$, then $\mathrm{proj}_W \mathbf{w} = \mathbf{w}$?

Solution: The definition of $\mathrm{proj}_W \mathbf{w}$ is written out in Theorem 8 on page 390 of the textbook. Once you notice that $\mathbf{u}_i\cdot\mathbf{u}_i = 1$ for all i, this definition coincides exactly with the right hand side of the equation in part (a). ♣

19. Consider the system $\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 2 & 2 \end{bmatrix}\mathbf{x} = \begin{pmatrix} 3 \\ 3 \\ 6 \end{pmatrix}$.

(a) Are there solutions to this system?

Solution: No, there aren't. You can check this by row reduction. ♣

(b) Find all least squares solutions to the system.

Solution: The first step is to compute the least-squares system $A^T A\mathbf{x} = A^T\mathbf{b}$:

$$A^T A = \begin{bmatrix} 9 & 14 \\ 14 & 24 \end{bmatrix} \qquad\qquad A^T \begin{pmatrix} 3 \\ 3 \\ 6 \end{pmatrix} = \begin{pmatrix} 21 \\ 30 \end{pmatrix}$$

So to find least squares solutions, we need to find solutions to the 2×2 system:

$$\begin{bmatrix} 9 & 14 \\ 14 & 24 \end{bmatrix} \mathbf{x} = \begin{pmatrix} 21 \\ 30 \end{pmatrix}$$

A bit of row reduction shows that there is only one solution to the 2×2 system, and therefore only one least squares solution to the original system. That solution is:

$$\mathbf{x} = \frac{1}{5} \begin{pmatrix} 21 \\ -6 \end{pmatrix}$$