MATH 46, REVIEW FOR FINAL

1. (a) Let $A \in M_{m\times n}$, $A = [a_{ij}]$. Define the following terms: *cofactor* of a_{ij}, $\det A$, *eigenvalue* of A, *eigenvector* associated to an eigenvalue of A, eigenspace associated to eigenvalue of A, A is *diagonalizable*.
   (b) For a function $T : V \to V$ define the following terms: T is a *linear transformation*, T is diagonalizable.
   (c) Describe the least squares method to solve $A\mathbf{x} = \mathbf{b}$. When might you use it to solve $A\mathbf{x} = \mathbf{b}$?
   (d) What definitions do you think will be on the test?

2. Let $A \in M_{m\times n}$. Assume for all $\mathbf{x} \in \mathbb{R}^n$ that $A\mathbf{x} = \mathbf{0}$. Prove that A is the zero matrix.

3. Solve the following linear system: $\begin{aligned} 2x + y - 2z &= 10 \\ 3x + 2y + 2z &= 1 \\ 5x + 4y + 3z &= 4 \end{aligned}$
   (a) by row reduction.
   (b) by Cramer's Rule. [NOT COVERED IN 2018]

4. Let A be an $m \times n$ matrix and let $T_A : \mathbb{R}^n \to \mathbb{R}^m$ be defined by $T_A(\mathbf{x}) = A\mathbf{x}$. Are the following statements true or false. If true give a proof, if false explain why.
   (a) $\dim \operatorname{Nul} A \leq n$.
   (b) $\operatorname{rank} A \leq m$.
   (c) If $n > m$ then the linear transformation T_A cannot be one-to-one.
   (d) If $n < m$ then T_A cannot be onto.
   (e) If T_A is one-to-one and $m = n$ then T_A must be onto.

5. Let A be an $n \times n$ matrix satisfying $A^3 = I_n$, where I_n is the $n \times n$ identity matrix. Show that $\det A = 1$.

6. For the following problems, prove the statement or give a specific counterexample:
   (a) $W = \{p \in P_2 \mid (p(3))^2 + p(3) = 0\}$ is a subspace of P_3.
   (b) $W = \{(x, y) \in \mathbb{R}_2 \mid x + y = 0\}$ is a subspace of $\mathbb{R}_2$.
   (c) Let V be a vector space and let $f_1 : V \to \mathbb{R}$ and $f_2 : V \to \mathbb{R}$ be linear transformations. Define $T : V \to \mathbb{R}_2$ defined by $T(\mathbf{v}) = (f_1(\mathbf{v}), f_2(\mathbf{v}))$. T is linear.

7. Let $S = \left\{ \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 8 & 2 \\ 4 & 6 \end{bmatrix} \right\}$.
   (a) Decide whether S is independent.
   (b) Let $W = \operatorname{span} S$. Use the result of (a) to find a basis of W that is a subset of S. Find $\dim W$.
   (c) Determine whether $\begin{bmatrix} 5 & 1 \\ -1 & 9 \end{bmatrix} \in W = \operatorname{span} S$.

8. If you are given a square upper triangular matrix, how would you tell at a glance whether or not it is invertible? Explain your answer using determinants. How would you tell at a glance the eigenvalues of the matrix and their multiplicities?

9. Let V and W be vector spaces and let $T : V \to W$ be a linear transformation.
   (a) Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_\ell\}$ be a set of vectors that spans V. Prove that the set $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3), \ldots, T(\mathbf{v}_\ell)\}$ spans range T in W.
   (b) Assume T is one-to-one, and assume S is a basis of V. Prove the set $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3), \ldots, T(\mathbf{v}_\ell)\}$ is independent.
   (c) Show that $L : P_3 \to \mathbb{R}_3$ be defined by $L(p) = (p(0), p(1), p(3))$. Prove L is a linear transformation.
   (d) Use the result of (a) and the basis $\mathcal{B} = \{1, t, t^2, t^3\}$ of P_3 to find a spanning set of range L. Find a subset of this spanning set that is a basis of range L.

10. (a) Let $T: V \to W$ be a linear transformation. State the *Rank plus Nullity Theorem* for T.
    (b) Let $T: M_{2\times 2} \to \mathbb{R}^6$ be a linear transformation. If nullity $T \leq 2$ what are the possible values of rank T?
    (c) Let V be a finite dimensional vector space and let $T: V \to \mathbb{R}^5$ be a linear transformation. If T is onto and nullity $T = 3$ what is $\dim V$?
    (d) Let V be a finite dimensional vector space and let $T: V \to V$ be linear. Prove that if T is one-to-one then T is onto.

11. Let $A = \begin{bmatrix} -5 & -9 & -6 \\ 0 & -2 & 0 \\ 3 & 9 & 4 \end{bmatrix}$.
    (a) Decide whether A is diagonalizable If so, find a diagonal matrix D and an invertible matrix P such that $D = P^{-1}AP$.
    (b) If A is diagonalizable, use the result of (a) to find A^{10}.

12. Determine whether or not each of the following matrices, A, is diagonalizable. Justify your answer. If A is diagonalizable find a diagonal matrix similar to A.
    (a) $A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (b) $A = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

13. Let A be a 4×4 matrix with characteristic polynomial $(\lambda^2 - 1)(\lambda - 3)^2$. Then A is diagonalizable if and only if the eigenspace associated with the eigenvalue 3 has what dimension?

14. Let A be a matrix with eigenvalues 1, -2, and 4. Is there necessarily a matrix B such that $B^2 = A$? Prove your answer.

15. Define a linear transformation $T: P_2 \to P_2$ by

$$T(a + bt + ct^2) = (a - b + 3c) + (2b + c)t + 3ct^2$$

    for all $a + bt + ct^2 \in P_2$ and let $\mathcal{B} = \{1, t, t^2\}$ be the standard basis of P_2.
    (a) Find $A = [T]_{\mathcal{B}}$.
    (b) Determine whether T is diagonalizable. If yes, find a basis $\mathcal{C}$ of P_2 such that the matrix of T with respect to $\mathcal{C}$, $[T]_{\mathcal{C}}$, is a diagonal matrix, and find $[T]_{\mathcal{C}}$. If T is not diagonalizable explain why not.

16. Let $W = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \right\}$.
    (a) Use the Gram-Schmidt process (§6.4) find an orthonormal basis of W.
    (b) Find a basis of $W^{\perp}$.

17. Let $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}$. Show that $\mathcal{B}$ is an orthogonal set. Let $\mathbf{x} \in \mathbb{R}^3$. Find the orthogonal projection of $\mathbf{x}$ onto $W = \text{span}\,\mathcal{B}$.

18. Let $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ be an orthonormal set in $\mathbb{R}^n$. Prove S is independent.
    (a) Let $W = \text{span}\, S$ and let $\mathbf{w} \in W$. Prove that $\mathbf{w} = (\mathbf{w} \cdot \mathbf{u}_1)\mathbf{u}_1 + \cdots + (\mathbf{w} \cdot \mathbf{u}_k)\mathbf{u}_k$.
    (b) Why does the result of (a) show that, if $\mathbf{w} \in \text{span}\, S$, then $\text{proj}_W \mathbf{w} = \mathbf{w}$?

19. Consider the system $\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 2 & 2 \end{bmatrix} \mathbf{x} = \begin{pmatrix} 3 \\ 3 \\ 6 \end{pmatrix}$.
    (a) Are there solutions to this system?
    (b) Find all least-squares solutions to the system.