Math 70 | **TUFTS UNIVERSITY** | **May 3, 2013**
Linear Algebra | **Department of Mathematics** | **All sections**

Final Exam

Instructions: No notes or books are allowed. All calculators, cell phones, or other electronic devices **must** be turned off and put away during the exam. Unless otherwise stated, you **must show all work** to receive full credit. *You are required to sign your exam. With your signature you are pledging that you have neither given nor received assistance on the exam. Students found violating this pledge will receive an F in the course.*

Problem	Point Value	Points
1	10	
2	12	
3	14	
4	5	
5	6	
6	6	
7	8	
8	8	
9	12	
10	6	
11	8	
12	5	
	100	

1. (10 pts) **True/false questions.** For each of the statements below, decide whether it is true or false. Indicate your answer by shading the corresponding box. There will be no partial credit.

   (a) Let W be a subspace of $\mathbb{R}^n$. W and $W^\perp$ have no vector in common. $\boxed{\text{T}}$ $\boxed{\text{F}}$

   (b) If $A \in M_{n\times n}$ is similar to a diagonal matrix, then A has n distinct eigenvalues. $\boxed{\text{T}}$ $\boxed{\text{F}}$

   (c) The zero vector is contained in any eigenspace and is hence an eigenvector. $\boxed{\text{T}}$ $\boxed{\text{F}}$

   (d) Similar matrices have the same eigenvalues. $\boxed{\text{T}}$ $\boxed{\text{F}}$

   (e) It is possible for $A \in \mathbb{M}_{5\times 5}$ to have 5 complex eigenvalues. $\boxed{\text{T}}$ $\boxed{\text{F}}$

2. (12 points) Set $\mathbf{w}_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{w}_2 = \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$; then $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{v}\}$ is a basis of $\mathbb{R}^3$ (you don't have to verify this). Let W be the plane in $\mathbb{R}^3$ spanned by $\{\mathbf{w}_1,\ \mathbf{w}_2\}$.

   (a) Verify that $\{\mathbf{w}_1, \mathbf{w}_2\}$ is an orthogonal set.

   (b) Verify that $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{v}\}$ is *not* an orthogonal set.

   (c) Find a vector $\widehat{\mathbf{v}}$ in W and a vector $\mathbf{x}$ in $W^\perp$ such that $\mathbf{v} = \widehat{\mathbf{v}} + \mathbf{x}$.

   (d) Find a vector $\mathbf{w}_3$ such that $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is an orthogonal basis of $\mathbb{R}^3$.

3. (14 points) Let $A = \begin{bmatrix} 2 & 0 & 1 & 2 \\ 1 & -1 & 0 & 0 \\ 0 & 2 & 1 & 2 \end{bmatrix}$.

   (a) Find the solution set to the matrix equation $Ax = 0$. Express your answer in **vector parametric form**.

   (b) Find a basis for Nul(A).

   (c) Give the dimension of Col(A) (justify your answer).

   (d) Do the columns of A span $\mathbb{R}^3$? Why or why not.

(e) Consider the linear transformation $T(x) = Ax$. Is T onto? Why or why not.

(f) Is T one-to-one? Why or why not.

4. (5 points) Using any appropriate method, find the inverse of the matrix $A = \begin{bmatrix} 5 & -2 \\ -3 & 4 \end{bmatrix}$.

5. (6 points) Given that A, B, C are $n \times n$ matrices with $\det(A) = -1$, $\det(B) = 2$, and $\det(C) = 4$, find the following:

(a) $\det(ABC)$

(b) $\det(B^T C^T)$

(c) $\det(C^{-1} B)$

6. (6 points) Let $V = \mathbb{M}_{2\times 2}$ be the set of all 2×2 matrices. Let H be the subset of all matrices in V which have rank less than or equal to 1. Is H a subspace of V? If your answer is "yes", give a proof of your assertion. If your answer is "no", give a counterexample that shows H cannot be a subspace of V.

7. (8 points) Let u_1, u_2, u_3 be 3 linearly independent vectors in $\mathbb{R}^5$, and let $H = \text{span}\{u_1, u_2, u_3\}$. Let $W = \text{span}\{u_1, u_2, u_1 + u_2 + u_3\}$. Show that $W = H$ by completing the following.

   (a) Show that an arbitrary element in H must be an element of W.

   (b) Show that an arbitrary element of W must be an element of H.

8. (8 points) Let $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \\ 1 & 2 \end{bmatrix}$ and $b = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}$.

   (a) Verify that the system $A\mathbf{x} = \mathbf{b}$ is inconsistent.

   (b) Compute the least squares solution to $A\mathbf{x} = \mathbf{b}$.

9. (12 points) Let V and W be vector spaces and let $T : V \to W$ be a linear transformation.
   (a) Using set notation, give the definition of ker(T).

   (b) Prove that ker(T) is a subspace of V.

10. (6 points) Suppose $T : \mathbb{P}_3 \to \mathbb{P}_3$ is linear and $\mathcal{B} = \{1, t, t^2\, t^3\}$.

    Given that $[T]_{\mathcal{B}} = \begin{bmatrix} 2 & 0 & -1 & 1 \\ 1 & 0 & 1 & 1 \\ 2 & -3 & 0 & 2 \\ 1 & 1 & 1 & 0 \end{bmatrix}$, find $T(3 - 2t + t^3)$.

11. (8 points)Let A be a 4×4 matrix with 4 distinct eigenvalues. One of those eigenvalues is 0.
    (a) Is A diagonalizable? Why or why not.

    (b) Is A invertible? Explain your reasoning.

12. (5 points) Let A be a non-zero square matrix (which is not the identity matrix) such that $A^2 = A$. Show that the eigenvalues of A are either 0 or 1.

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Name ____

Instructor ____________________________________

I pledge that I have neither given nor received assistance on this exam.

Signature _____________________________________