Math 46 - 01 | Tufts University
Department of Mathematics
Final Exam | December 13, 2007

No books, notes, or calculators are allowed on the exam. Remember to sign your exam book. With your signature, you are pledging that you have neither given nor received help in this exam.

1. (10 points) Let $A = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$. The eigenvalues of A are 1 and -2.
   Find an *orthogonal* matrix U such that $U^T AU$ is diagonal.

2. (8 points) Apply the Gram-Schmidt Process to the following set of vectors:
$$u_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

3. (15 points) Let $u_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $u_2 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$ and let $W = \text{span}\,\{u_1, u_2\}$. Let $y = \begin{pmatrix} 0 \\ 5 \\ -4 \end{pmatrix}$.
   (a) Find the point w in W nearest to y.
   (b) Find $z \in W^\perp$ such that $y = w + z$.
   (c) Find dist (W, y).
   (d) The vectors u_1, u_2, z are linearly independent. How can you see this without calculating a determinant or row reducing?
   (e) Find the unit vector in the direction of u_1.

4. (5 points) Let $A = \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}$. Find an eigenvector of A in $\mathbf{C}^2$.

5. (5 points) Fill in the blanks: Let $f(x) = (x - \lambda_1)^{m_1} \cdots (x - \lambda_d)^{m_d}$ be the characteristic polynomial of a matrix A. Then A is diagonalizable if for each $i = 1, 2, \ldots, d$, the dimension of ______ equals ______.

6. (12 points) Let $\mathcal{B} = \{v_1, v_2\}$ be an ordered basis of a vector space V. Suppose $T : V \to V$ is a linear transformation such that $T(v_1) = -v_1$ and $T(v_2) = v_1 + v_2$.
   (a) Find $[T]_{\mathcal{B}}$.
   (b) Find x_1, x_2 not both 0 such that $T(x_1 v_1 + x_2 v_2) = x_1 v_1 + x_2 v_2$.
   (c) Find a basis $\{u_1, u_2\}$ of V such that both u_1 and u_2 are eigenvectors of T.

Exam continues on the other side.

7. (14 points) Let $A = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$, let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T(w) = Aw$ for all $w \in \mathbb{R}^2$, let $\mathcal{B} = \{v_1, v_2\}$, where $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$ and let

$$P = \ P\ ,$$

the change-of-basis matrix from the ordered basis $\mathcal{B}$ to the standard basis $\mathcal{S}$ of $\mathbb{R}^2$.

(a) Find P.
(b) Find the change-of-basis matrix $\ P$.
(c) Find $T(v_1)$ and $T(v_2)$.
(d) Find $[T(v_1)]_{\mathcal{B}}$ and $[T(v_2)]_{\mathcal{B}}$.
(e) Find $[T]_{\mathcal{B}}$.
(f) Fill in the blank: $[T]_{\mathcal{B}}[w]_{\mathcal{B}} =$ ______ for all $w \in \mathbb{R}^2$.
(g) Either $P^{-1}AP$ or PAP^{-1} equals $[T]_{\mathcal{B}}$. Which is correct?

8. (12 points) Give the definitions:
(a) A is similar to B.
(b) v is an eigenvector of A.
(c) The map $T : V \to W$ is one-to-one.
(d) The set $v_1, \ldots, v_p$ is linearly independent.

9. (9 points) Let A be $n \times n$. Give three distinct assertions about A which are equivalent to the assertion that $\det(A) \neq 0$.

10. (10 points) Suppose A is a symmetric matrix. Let v_1 and v_2 be eigenvectors of A corresponding to <u>*distinct*</u> eigenvalues λ_1 and λ_2. Fill in the blanks:

$$\begin{aligned}
\lambda_1(v_1 \cdot v_2) &= (\lambda_1 v_1) \cdot v_2 \text{ because } \underline{(a)\qquad\quad} \\
&= Av_1 \cdot v_2 \\
&= (Av_1)^T v_2 \\
&= ({v_1}^T A^T) v_2 \text{ because } \underline{(b)\qquad\quad} \\
&= ({v_1}^T A) v_2 \text{ because } \underline{(c)\qquad\quad} \\
&= {v_1}^T (Av_2) \text{ because } \underline{(d)\qquad\quad} \\
&= {v_1}^T (\lambda_2 v_2) \\
&= \lambda_2 (v_1 \cdot v_2)
\end{aligned}$$

Therefore $\underline{(e)\qquad\quad}$.

End of Exam