Math 46

Tufts University
Department of Mathematics
Final Exam

December 14, 2005

1. Let $A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$.

   (a) (2 points) Verify that the system $A\mathbf{x} = \mathbf{b}$ is inconsistent.

   (b) (10 points) Find the least-squares solution to $A\mathbf{x} = \mathbf{b}$ in two ways: one that utilizes $\text{proj}_{col(A)}\mathbf{b}$ and one that utilizes the normal equations. Clearly specify which one is which.

   (c) (2 points) Compute the least-square error of your solution from part (b).

2. Let $A = \begin{bmatrix} 1 & 2 & -9 & 2 \\ 0 & 1 & 6 & 4 \\ 1 & 2 & -6 & 4 \end{bmatrix}$.

   (a) (2 points) Do the columns of A span $\mathbb{R}^4$? Why or why not?

   (b) (2 points) Do the columns of A span $\mathbb{R}^3$? Why or why not?

   (c) (2 points) What is the rank of A?

   (d) (5 points) Find $Nul(A)$. Write your answer in set notation.

   (e) (1 point) Check that the vectors found in (d) really are in $Nul(A)$.

3. Let $T : \mathbf{P}_2 \rightarrow \mathbb{R}^3$ be the linear transformation given by $T(\mathbf{p}(t)) = \begin{bmatrix} \mathbf{p}(-1) \\ \mathbf{p}(0) \\ \mathbf{p}'(-1/2) \end{bmatrix}$.

   (a) (2 points) $Ker(T)$ is a subspace of ______.

   (b) (5 points) Find the kernel of T. Write your answer in set notation.

   (c) (4 points) Let $\mathcal{B} = \{1, t, t^2\}$. Find the standard matrix of T relative to $\mathcal{B}$ and the standard basis for $\mathbb{R}^3$.

   (d) (1 point) In *one* computation, check your answers to (b) and (c).

4. Let

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 3 \\ 4 \\ -7 \end{bmatrix}, \quad \mathbf{a}_3 = \begin{bmatrix} -5 \\ -8 \\ 9 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 4 \\ 7 \\ -6 \end{bmatrix}.$$

   (a) (5 points) Is $\mathbf{b}$ in $span\{\mathbf{a}_1,\ \mathbf{a}_2,\ \mathbf{a}_3\}$?

   (b) (5 points) Could $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ be used as a basis for $\mathbb{R}^3$? Why or why not?

5. Let $\mathbf{p}_1(t) = 1 + t^2, \quad \mathbf{p}_2(t) = t, \quad \mathbf{p}_3(t) = 1 + t.$

   (a) (5 points) Verify that $\mathcal{B} = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is a basis of $\mathbf{P}_2$.

   (b) (3 points) Find $[3t + 4t^2]_{\mathcal{B}}$.

6. Consider a homogeneous system of 12 linear equations in 14 unknowns whose matrix form is $A\mathbf{x} = \mathbf{0}$.

   (a) (2 points) What is the size of the coefficient matrix A?

(b) (4 points) What is the largest that rank (A) can be?

(c) (4 points) Is it possible that all solutions of $A\mathbf{x} = \mathbf{0}$ are multiples of one fixed nonzero vector? Why or why not?

7. (4 points) Let $W = \left\{ \begin{bmatrix} a & b \\ c & 0 \\ c & 0 \end{bmatrix} : \ a, b, c \in \mathbb{R} \right\}$. Show that W is a subspace of $M_{3\times 2}$.

8. Suppose A is a 6×4 matrix whose reduced echelon form has 4 pivots. Let T be the linear transform defined by $T(\mathbf{x}) = A\mathbf{x}$.

(a) (3 points) What are the domain and codomain of T?

(b) (3 points) Is T a one-to-one mapping? Why or why not?

(c) (3 points) Do the columns of A span the codomain of T? Why or why not?

9. (a) (2 points) Given three square matrices A, D, and P with P invertible, show that $A = PDP^{-1}$ if and only if $AP = PD$.

(b) (5 points) Suppose A is the matrix $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$. Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

(c) (1 point) Check your answer to (b) by verifying that $AP = PD$.

10. (1 point each) Indicate whether each of the following statements is true or false. You do not need to give explanations as there is no partial credit on this problem.

(a) If A is an $m \times n$ matrix and A has only the zero vector in its null space, then $rank(A) = m$.

(b) If A is an $n \times n$ matrix and one of the eigenvalues of A is 0, then A is not invertible.

(c) If A is an $n \times n$ matrix with $n - 1$ distinct eigenvalues, then A is not diagonalizable. Assume all entries of A and all eigenvalues are real.

(d) If A is an $m \times n$ matrix and $rank(A) = n$, then the system $A\mathbf{x} = \mathbf{b}$ has a solution $\forall\ \mathbf{b} \in \mathbb{R}^m$.

(e) If $\hat{\mathbf{x}}$ is a least squares solution to $A\mathbf{x} = \mathbf{b}$, then the least square error is the distance from $\mathbf{b}$ to $\hat{\mathbf{x}}$.

(f) Let A be an $m \times n$ matrix with $m > n$ and let $\mathbf{b} \in \mathbb{R}^m$. Then there is a unique solution to the least squares problem $A\mathbf{x} = \mathbf{b}$.

(g) If A can be row-reduced to B, then A and B have the same eigenvalues.

(h) Every basis for a vector space V contains the same number of elements.

11. (5 points) Suppose $T : V \to W$ is a linear transformation between vector spaces V and W. Prove that if $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a linearly **dependent** set in V, then $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_k)\}$ is a linearly **dependent** set in W.