MATH 70 SECTION 01 FINAL EXAM
FALL 2017

SOLUTIONS

(1) (5+10=15 points) Short answer questions: no partial credit, and no work or explanations required:

True or False? (circle your answers)

(a) Suppose A is a 5×3 matrix which has 3 pivots. Let T be the linear transformation defined by $T(\vec{x}) = A\vec{x}$. T is not onto. [T] F

(b) Every linearly independent set in $\mathbb{R}^n$ is an orthogonal set. T [F]

(c) If the dimension of the vector space V is p for some $p \geq 1$, then every set of vectors that spans V has more than p vectors. T [F]

(d) There exists a one-to-one linear transformation from $\mathbb{P}_3$ to $\mathbb{R}^3$. T [F]

(e) Suppose A is an $m \times n$ matrix. Then Nul$A$ is orthogonal to Col$A$. T [F]

Short Answer

(a) Suppose U is a square matrix with orthonormal columns. Explain why U is invertible using theorems from the class.

> **Solution:**
>
> Since $U^T U = I$ then and U is square, the invertible matrix theorem give us $UU^T = I$ so U^T must be the inverse of U.

(b) Suppose a 8×6 matrix A has 4 pivot columns. What is the dimension of Nul$A$?

> **Solution:**
>
> The dimension of the null space of A is equal to the number of free non-pivot columns of A, so in this case must be 2.

(c) Suppose W is a subspace of $\mathbb{R}^n$. If I take the union of orthogonal bases for W and $W^\perp$, why does this set span $\mathbb{R}^n$?

> **Solution:**
>
> First, all the vectors in the union of the bases are orthogonal, the ones in the basis for W are orthogonal to each other as are the ones in the basis for $W^\perp$ because the basis are orthogonal and each in W is orthogonal to all vectors in $W^\perp$ as the vector spaces are orthogonal. And because all the vectors are orthogonal they are also linearly independent.

Second, each vector $\vec{x}$ in $\mathbb{R}^n$ can be written as $\vec{x} = \text{proj}_W \vec{x} + \vec{z}$ where $\vec{z}$ is in $W^\perp$ by the orthogonal decomposition theorem. Therefore the union of the basis for W and $W^\perp$ forms a basis for R^n.

(d) Gram-Schmidt is an algorithm for doing what?

Solution:

Gram-Schmidt is a process that takes a given basis for a vector space and finds and orthogonal basis for the same space.

(e) Suppose A, B are both $n \times n$ matrices for some n. Show that if A is similar to B, then A^2 is similar to B^2.

Solution:

Since A is similar to B, there must be an invertible matrix P such that $A = PBP^{-1}$. This means that

$$A^3 = (PBP^{-1})^3 = PBP^{-1}PBP^{-1}PBP^{-1} = PB^3P^{-1}$$

so A^3 is similar to B^3.

Questions 2-8 have partial credit, and work/explanations/justifications ARE required:

(2) (2+4+4=10 pts) Consider the matrix A below.

$$\mathcal{A} = \begin{pmatrix} 3 & 1 \\ 1 & -1 \\ -1 & 1 \\ 1 & -1 \end{pmatrix}$$

(a) Show that the columns of A are orthogonal.

Solution:

$$\begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = 3(1) + 1(-1) + -1(1) + 1(-1) = 0$$

(b) Show that the vector $\vec{y} = \begin{bmatrix} 3 \\ 1 \\ 5 \\ 1 \end{bmatrix}$ is *not* in Col$A$.

Solution:

$$[A|\vec{y}] =$$

$$\begin{bmatrix} 3 & 1 & 3 \\ 1 & -1 & 1 \\ -1 & 1 & 5 \\ 1 & -1 & 1 \end{bmatrix} \quad R_4 - R_2 = R_4$$

$$\begin{bmatrix} 3 & 1 & 3 \\ 1 & -1 & 1 \\ -1 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix} \quad 1R_2 + R_3 = R_3$$

$$\begin{bmatrix} 3 & 1 & 3 \\ 1 & -1 & 1 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix} \quad R_1 - 3R_2 = R_1$$

$$\begin{bmatrix} 0 & 4 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

Notice that row 3 indicates that there is no solution to $A\vec{x} = \vec{y}$ meaning y is not in the Col(A).

(c) Find the vector $\hat{y}$ in Col$A$ that is closest to $\vec{y}$.

Solution:

$$\mathcal{A} = \begin{bmatrix} 3 & 1 \\ 1 & -1 \\ -1 & 1 \\ 1 & -1 \end{bmatrix} \quad \mathcal{A}^T = \begin{bmatrix} 3 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

$$[A^T A | A\vec{y}] = \begin{bmatrix} 12 & 0 & 6 \\ 0 & 4 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 3/2 \end{bmatrix}$$

This gives us that $A\hat{x} = A \begin{bmatrix} 1/2 \\ 3/2 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 1 \\ -1 \end{bmatrix}$ is the closest vector in $\mathrm{Col}(A)$ to $\vec{y}$.

(3) (2+4+2=10 pts) Consider the matrix A below.

$$A = \begin{pmatrix} 4 & 2 & 3 & 3 \\ 0 & 2 & h & 3 \\ 0 & 0 & 4 & 14 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

(a) What are the 4 eigenvalues of A? (Note this does not depend on what the value of h is!)

Solution:

Since this is an upper triangular matrix the eigenvalues are on the diagonal. So the eigenvalues are 4 and 2.

(b) What value of h will make the eigenspace for $\lambda = 4$ two dimensional?

Solution:

The eigenspace for $\lambda = 4$ is $\mathrm{Nul}(A - 4I)$. Since

$$A - \lambda I = \begin{bmatrix} 0 & 2 & 3 & 3 \\ 0 & -2 & h & 3 \\ 0 & 0 & 0 & 14 \\ 0 & 0 & 0 & -2 \end{bmatrix} \quad \tfrac{R_3}{14} \leftrightarrow R_3$$

$$\begin{bmatrix} 0 & 2 & 3 & 3 \\ 0 & -2 & h & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix} \quad -\tfrac{R_4}{2} = R_4$$

$$\begin{bmatrix} 0 & 2 & 3 & 3 \\ 0 & -2 & h & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R_4 - R_3 = R_4$$

$$\begin{bmatrix} 0 & 2 & 3 & 3 \\ 0 & -2 & h & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad 1R_1 + R_2 = R_2$$

$$\begin{bmatrix} 0 & 2 & 3 & 3 \\ 0 & 0 & h+3 & 6 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

for the eigenspace for $\lambda = 4$ to have dimension 4 h must equal 3.

(c) Suppose you put this value of h in A. What would you do next to decide whether A was diagonalizable or not? In particular, what would need to be true for A to be diagonalizable?

Solution:

We next check the dimension of the eigenspace for $\lambda = 2$, for A to be generalizable we need it to have dimension 2.

(4) (2+3+3+4=12 pts) Let $M_{2\times 2}$ be the vector space of 2×2 real matrices with real entries. Consider the transformation $f : M_{2\times 2} \to \mathbb{R}^2$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{bmatrix} 3a+b \\ c+d \end{bmatrix}$$

(a) Show that f is linear.

Solution:

Notice that

$$f\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix}\right) = f\left(\begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}\right) = \begin{bmatrix} 3(a+e)+b+f \\ c+g+d+h \end{bmatrix}$$
$$= \begin{bmatrix} 3a+b \\ c+d \end{bmatrix} + \begin{bmatrix} 3e+f \\ g+h \end{bmatrix} = f\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) + f\left(\begin{bmatrix} e & f \\ g & h \end{bmatrix}\right)$$

And

$$f\left(e\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = f\left(\begin{bmatrix} ea & eb \\ ec & ed \end{bmatrix}\right) = \begin{bmatrix} e(3a+b) \\ e(c+d) \end{bmatrix} =$$
$$e\begin{bmatrix} 3a+b \\ c+d \end{bmatrix} = ef\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$$

This shows f is linear.

(b) Find a matrix for the linear transformation f in terms of the basis:

$$\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$$

for $M_{2\times 2}$ and the standard basis $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$ for $\mathbb{R}^2$.

Solution:

$$A = \left[f\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) f\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) f\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) f\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)\right] = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Note that, if A is a 2×2 matrix, then the coordinates of $f(A)$ in the standard basis of $\mathbb{R}^2$ *is* the vector $f(A)$. Normally, you would take the coordinates of f of the $\mathcal{B}$ basis in the $\mathcal{C}$ basis to find ${}_{\mathcal{C}}[f]_{\mathcal{B}}$.

(c) What does it mean for a transformation T to be one-to-one?

Solution:

T is one-to-one iff for all $\vec{x}$ and $\vec{y}$ in the domain of T,, $T(\vec{x}) = T(\vec{y})$ implies $\vec{x} = \vec{y}$.

(d) Either prove f as above is one-to-one, or find specific matrices that show it is not.

Solution:

Notice that $\det\left(\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}\right) \neq 0$. So$A$ is an invertible matrix, meaning the matrix transformation $\vec{x} \to A\vec{x}$ is one-to-one ans so T must be also.

(5) (2+4+2=8 pts) Let W be the subspace with basis $\vec{v_1} = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$ and $\vec{v_2} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$.

(a) Verify that $\vec{v_1}$ and $\vec{v_2}$ are NOT orthogonal.

Solution:

$$\vec{v_1} \cdot \vec{v_2} = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = 3 + 12 + 0 = 15.$$

(b) Find an orthogonal basis for W by replacing $\vec{v_2}$ with vector a $\vec{u_2}$ that is orthogonal to $\vec{v_1}$ with $\text{Span}\{\vec{v_1}, \vec{u_2}\} = W$.

Solution:

$$\vec{u}_2 = \vec{v}_2 - \text{proj}_{\vec{v}_1}\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \tfrac{15}{45}\begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

(c) Suppose we let $\vec{v}$ we a vector that is not in W. Explain what I would do to find a vector $\vec{u_3}$ such that $\{\vec{u_1}, \vec{u_2}, \vec{u_3}\}$ is an orthogonal basis for $\mathbb{R}^3$. Draw a schematic diagram if that helps!

Solution:

Let $\vec{u}_1 = \vec{v}_1$ and $\vec{u}_2$ be the vector calculated in part (b) and $W = \text{Span}\,\{\vec{u}_1, \vec{u}_2\}$ define $\vec{u}_3 = \vec{v} - \text{proj}_W\vec{v}$

(6) (6 pts) Suppose B is the reduced echelon form for the matrix A.

$$\mathcal{A} = \begin{pmatrix} 1 & 2 & -4 & 3 & 3 \\ 5 & 10 & -9 & -7 & 8 \\ 4 & 8 & -9 & -2 & 7 \\ -2 & -4 & 5 & 0 & -6 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 & 0 & -5 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(a) Find a basis for Nul A.

Solution:

$$\mathcal{B} = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} \right\}$$

(b) Find a basis for Col A.

Solution:

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 5 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} -4 \\ -9 \\ -9 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 8 \\ 7 \\ -6 \end{bmatrix} \right\}$$

(c) Let $\vec{b} = \begin{bmatrix} 2 \\ 8 \\ 4 \\ -17 \end{bmatrix}$. Suppose $\begin{bmatrix} -1 \\ 0 \\ 3 \\ 0 \\ 5 \end{bmatrix}$ is a solution to the equation $A\vec{x} = \vec{b}$. Describe the solution set to $A\vec{x} = \vec{b}$ in parametric form.

Solution:

All solutions have the form $\begin{bmatrix} -1 \\ 0 \\ 3 \\ 0 \\ 5 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 5 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}$ where s and t are scalars.

(7) (9 pts) For each of the following give an example of a matrix with the stated property. EXPLAIN why your examples work.

(a) Find a 2×2 matrix that is invertible but not diagonalizable.

Solution:

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

(b) Find a 2×2 matrix that is diagonalizable but not invertible.

Solution:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

(c) Find a 2×3 matrix A NOT in reduced echelon form such that the mapping $\vec{x} \mapsto A\vec{x}$ is *not* onto.

Solution:

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

(8) (6+2=8 pts) Consider the matrix A given here:

$$A = \begin{pmatrix} 1 & -6 \\ 2 & -6 \end{pmatrix}$$

(a) Diagonalize the matrix A. That is, find matrices P, D with $A = PDP^{-1}$.

Solution:

$\det(A-\lambda I) = (1-\lambda)(-6-\lambda)+12 = (\lambda+3)(\lambda+2)$ so -3 and -2 are eigenvalues. The eigenspace for $\lambda = -3$ is $\text{Nul}(A+3I) = \text{Span}\left\{\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right\}$ and the eigenspace for $\lambda = -2$ is $\text{Nul}(A+2I) = \text{Span}\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$.

So $P = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix}$

(b) Use your answer from the previous part to *explain how you would* compute A^{37}

Solution:

$A^{37} = (PDP^{-1})^{37} = PD^{37}P^{-1} = P\begin{bmatrix} (-3)^{37} & 0 \\ 0 & (-2)^{37} \end{bmatrix} P^{-1}$

(9) (2+2+6=10 pts) Suppose W is a subspace of $\mathbb{R}^n$. Consider the set $W^\perp$.

(a) What does it mean for the a vector $\vec{z}$ from $\mathbb{R}^n$ to be in $W^\perp$?

Solution:

It means that $\vec{x}$ is orthogonal to all vectors in W.

(b) What do you need to prove to show $W^\perp$ is a subspace of $\mathbb{R}^n$.

Solution:

We must show it contains the zero vector, is closed under vector addition and closed under scalar multiplication.

(c) Show that $W^\perp$ is a subspace of $\mathbb{R}^n$.

Solution:

(i) Let $\vec{v} \in W$, since $\vec{0} \cdot \vec{v} = 0$ then $\vec{0} \in W^\perp$.
(ii) Let $\vec{v}, \vec{u} \in W^\perp$ and $\vec{w} \in W$, then $\vec{w} \cdot (\vec{v} + \vec{u}) = \vec{w} \cdot \vec{v} + \vec{w} \cdot \vec{u} = 0 + 0 = 0$ therefore $(\vec{v} + \vec{u}) \in W^\perp$.
(iii) Let $\vec{v} \in W^\perp$, c a scalar and $\vec{w} \in W$, then $\vec{w} \cdot (c\vec{v}) = c(\vec{w} \cdot \vec{v}) = 0$. So, $c\vec{v} \in W^\perp$.

(10) (2+6=8 pts) Let W and U be subspaces of a vectors space V. Suppose the intersection, $W \cap U$, of W and U contains only the zero vector $\vec{0}$. Let $\{\bar{w}_1, \dots, \bar{w}_p\}$ and $\{\bar{u}_1, \dots, \bar{u}_k\}$ be bases of W and U, respectively.

(a) What does it mean for the set $\{\bar{w}_1, \dots, \bar{w}_p, \bar{u}_1, \dots, \bar{u}_k\}$ to be linearly independent - i.e. give the definition of linear independence of this set.

Solution:

For $\{\bar{w}_1, \dots, \bar{w}_p, \bar{u}_1, \dots, \bar{u}_k\}$ to be linearly independent, the only way for

$$c_1\bar{w}_1, + \cdots + c_p\bar{w}_p + d_1\bar{u}_1 + \cdots + d_k\bar{u}_k = 0$$

is if all the c_i's and d_j's are zero.

(b) Show that $\{\bar{w}_1, \dots, \bar{w}_p, \bar{u}_1, \dots, \bar{u}_k\}$ is linearly independent.

Solution:

Suppose to wards a contradiction that

$$c_1\bar{w}_1, + \cdots + c_p\bar{w}_p + d_1\bar{u}_1 + \cdots + d_k\bar{u}_k = 0$$

where some c_i's or d_j's are not zero. Then

$$c_1\bar{w}_1, + \cdots + c_p\bar{w}_p + d_1\bar{u}_1 + \cdots + d_k\bar{u}_k = 0$$

so

$$c_1\bar{w}_1, + \cdots + c_p\bar{w}_p = -d_1\bar{u}_1 + \cdots - d_k\bar{u}_k$$

Let $\vec{x} = c_1\bar{w}_1, + \cdots + c_p\bar{w}_p = -d_1\bar{u}_1 + \cdots - d_k\bar{u}_k$ This means that $\vec{x}$ must be in W as it is a linear combination of the basis vectors for W, but also $\vec{x}$ must be in U as it is equal to a linear combination of basis vectors for U. Since $W \cap U$ contains only the zero vector we have that $\vec{x} = \vec{0}$ but this is a contradiction since there must be at least one c_i or d_j that is nonzero.

Math 70-01 **Final Exam** **Fall 2017**

This version of the exam is for SECTION 01, taught by Professor Walsh, which meets Tues/Wed/Fri at 9:30. Make sure you have the right exam.

Name: ____

I pledge that I have neither given nor received assistance on this exam.

Signature ____