KEY

Math 70
Linear Algebra

TUFTS UNIVERSITY
Department of Mathematics
Exam II

March 30, 2015
Sections 1 and 2

Instructions: No notes or books are allowed. All calculators, cell phones, or other electronic devices **must** be turned off and put away during the exam. Unless otherwise stated, you **must show all work** to receive full credit. *You are required to sign your exam. With your signature you are pledging that you have neither given nor received assistance on the exam. Students found violating this pledge will receive an F in the course.*

Problem	Point Value	Points
1	10	
2	8	
3	16	
4	12	
5	8	
6	8	
7	10	
8	10	
9	10	
10	8	
	100	

1. (10 pts) **True/false questions.** Decide whether each of the statements below is true or false. Indicate your answer by shading the appropriate box. No explanations needed. No partial credit.

   (a) For square matrices A and B, provided AB and BA are defined, $|AB| - |BA| = 0$. $\boxed{\text{T}}$ $\boxed{\text{F}}$

   (b) For a square invertible matrix A, $\det(A^{-1}) = \frac{1}{\det A}$. $\boxed{\text{T}}$ $\boxed{\text{F}}$

   (c) For any square matrix A, if $|A| \neq 0$ then A is row equivalent to I. $\boxed{\text{T}}$ $\boxed{\text{F}}$

   (d) If A is an invertible matrix, then for any square matrix B for which AB is defined, AB is also invertible. $\boxed{\text{T}}$ $\boxed{\text{F}}$

   (e) If A and B are square matrices and $AB = B$, then A = I. $\boxed{\text{T}}$ $\boxed{\text{F}}$

2. (10 pts) It is a fact that $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 1 & -3 \\ 0 & 1 & -1 \end{bmatrix}$ are inverses of each other. Use this fact to solve the system of equations:

$$\begin{array}{rcrcrcl} & - & y & + & 2z & = & 2 \\ x & + & y & - & 3z & = & 5 \\ & & y & - & z & = & 2 \end{array}$$

$$\begin{bmatrix} 0 & -1 & 2 \\ 1 & 1 & -3 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix}$$

$$B \quad \vec{x} \quad = \quad b$$

$$\vec{x} = B^{-1} b = A b$$

$$= \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 11 \\ 6 \\ 4 \end{bmatrix}$$

$x = 11$
$y = 6$
$z = 4$

Check: $0 - 6 + 2(4) = 2 \checkmark$

$11 + 6 - 3(4) = 5 \checkmark$

$0 + 6 - 4 = 2 \checkmark$

3. (16 pts) Let V and W be vector spaces.

(a) Let $T : V \to W$ be a transformation. Give the conditions from the definition for T to be a *linear* transformation.

① $T(u+v) = T(u) + T(v)$ for all $u, v \in V$.

② $T(cu) = cT(u)$ for all $u \in V$, scalars c.

(b) Define the transformation $T : \mathbb{M}_{2\times 2} \to \mathbb{P}_2$ by $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + dt^2$.

(i) Use your definition from part (a) to prove T is a linear transformation.

① Let $u = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $v = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$ k a scalar

$$T(u+v) = T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}\right)$$

$$= T\left(\begin{bmatrix} a+a' & b+b' \\ c+c' & d+d' \end{bmatrix}\right) = (a+a') + (d+d')t^2$$

$$= (a + dt^2) + (a' + d't^2)$$

$$= T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) + T\left(\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}\right) \boxtimes$$

② $T\left(k\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = T\left(\begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}\right)$

$$= ka + (kd)t^2$$

$$= k(a + dt^2) = kT\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) \boxtimes$$

$$T : \mathbb{M}_{2\times 2} \to \mathbb{P}_2 \text{ by } T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + dt^2.$$

(ii) Find a basis for the kernel of T.

Suppose $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + dt^2 = 0$ for all t

Then $a = 0$ and $d = 0$. So b and c can be any scalars.

$\text{Ker } T = \left\{ \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix} : b, c \in \mathbb{R} \right\} = \text{Span}\left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$

Basis for Ker $T = \left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$

(These vectors are LI since neither is a multiple of the other.)

(iii) Find a basis for the range of T.

Range of $T = \{ a + dt^2 : a, d \in \mathbb{R} \} = \text{Span}\{1, t^2\}$

Basis for Range of $T = \{1, t^2\}$

(These vectors are LI since neither is a multiple of the other)

4. (10 pts) Consider the polynomials $\mathbf{p}_1(t) = 1 + t$, $\mathbf{p}_2(t) = 2 + t^2$, $\mathbf{p}_3(t) = 5 + 3t + t^2$.

(a) Is the set $S = \{\mathbf{p}_1(t), \mathbf{p}_2(t), \mathbf{p}_3(t)\}$ linearly independent in $\mathbb{P}_2$? Explain.

Since $\mathbb{P}_2 \cong \mathbb{R}^3$ we use the standard basis $\mathcal{B} = \{1, t, t^2\}$ for $\mathbb{P}_2$ and coordinate vectors to answer this:

$$\begin{array}{c} 1 \\ t \\ t^2 \end{array} \begin{bmatrix} [p_1(t)]_{\mathcal{B}} & [p_2(t)]_{\mathcal{B}} & [p_3(t)]_{\mathcal{B}} \end{bmatrix} = \overset{A}{\begin{bmatrix} 1 & 2 & 5 \\ 1 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix}} \begin{array}{l} \\ -R1 \\ \\ \end{array} \begin{bmatrix} 1 & 2 & 5 \\ 0 & -2 & -2 \\ 0 & 1 & 1 \end{bmatrix} \begin{array}{l} \\ \\ +2R2 \end{array}$$

$$\begin{bmatrix} 1 & 2 & 5 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{bmatrix},$$

Since column 3 is not a pivot column, $A\mathbf{x} = \mathbf{0}$ has free variables $\rightarrow$ columns of A are <u>not</u> L.I.

So by the isomorphism with $\mathbb{P}_2$, S is <u>not</u> L.I.

(b) Find a basis for the subspace of $\mathbb{P}_2$ spanned by S.

We know from (a) that S is not L.I. Since no 2 vectors in S are multiples of each other, we can choose any 2 to form a basis for Span S.

e.g. $\{p_1(t), p_2(t)\}$.

5. (8 pts) Let V be the vector space of all continuous, real-valued functions. Let H be the subspace of V with basis $\mathcal{B} = \{e^t, \cos(t), \sin(t), 1\}$.

(a) H is isomorphic to $\mathbb{R}^n$ where $n =$ 4 .

(b) Find $[3 + 2\cos(t) + 5e^t]_{\mathcal{B}}$.

$$= \begin{bmatrix} 5 \\ 2 \\ 0 \\ 3 \end{bmatrix}$$

6. (8 pts) Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ where $\mathbf{b}_1 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

(a) Find the vector $\mathbf{x}$ whose coordinate vector relative to $\mathcal{B}$ is given by: $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$

$$\underbrace{\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}}_{P_B} \underbrace{\begin{bmatrix} -2 \\ 5 \end{bmatrix}}_{[\vec{x}]_B} = \underbrace{\begin{bmatrix} -9 \\ 16 \end{bmatrix}}_{\vec{x}}$$

$$\left(\text{or just } -2\begin{bmatrix} 2 \\ -3 \end{bmatrix} + 5\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -9 \\ 16 \end{bmatrix}\right)$$

(b) Suppose $\mathbf{x} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$. Find the coordinates of $\mathbf{x}$ relative to $\mathcal{B}$ - i.e. find $[\mathbf{x}]_{\mathcal{B}}$.

$$[\vec{x}]_B = P_B^{-1}\vec{x}$$

$$= \frac{1}{1}\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 \\ 12 \end{bmatrix}$$

7. (10 pts) For $A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix}$, find a basis for each of the following spaces:

(a) Nul(A)

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix} \begin{matrix} \\ \\ -R1 \end{matrix} \sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} -3R2 \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 - x_3 = 0$ | $x_1 = x_3$

$x_2 + x_3 = 0$ | $x_2 = -x_3$

$x_3 = x_3$ | $x_3 = x_3$

Basis for Nul $A = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$

(b) Col(A)

Basis for col A = pivot columns of $\underline{A}$ $= \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} \right\}$

8. (10 pts) Compute $\begin{vmatrix} 1 & 0 & 0 & 2 \\ 0 & 3 & 4 & 5 \\ 5 & 4 & 0 & 3 \\ 2 & 0 & 0 & 1 \end{vmatrix}$ and use result to decide whether or not the matrix is invertible.

$$\begin{vmatrix} 1 & 0 & 0 & 2 \\ 0 & 3 & 4 & 5 \\ 5 & 4 & 0 & 3 \\ 2 & 0 & 0 & 1 \end{vmatrix} \begin{matrix} \\ \\ -5R1 \\ -2R1 \end{matrix} = \begin{vmatrix} 1 & 0 & 0 & 2 \\ 0 & 3 & 4 & 5 \\ 0 & 4 & 0 & -7 \\ 0 & 0 & 0 & -3 \end{vmatrix}$$

$$= 1 \begin{vmatrix} 3 & 4 & 5 \\ 4 & 0 & 7 \\ 0 & 0 & -3 \end{vmatrix} = 1(-3)\begin{vmatrix} 3 & 4 \\ 4 & 0 \end{vmatrix} = -3(-16) = 48$$

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

Since $48 \neq 0$ this matrix is invertible.

9. (12 pts)

(a) Let H be a subset of a vector space V. Give the conditions from the definition for H to be a subspace.

① $0_V \in H$

② $u+v \in H$ for all $u, v \in H$.

③ $cu \in H$ for all $u \in H$ and scalars c.

(b) Use the definition from part (a) to show that $\underbrace{\left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \,\middle|\, a+b=0 \right\}}_{H}$ is a subspace of $\mathbb{M}_{2\times 2}$.

① $0_{M_{2\times 2}} \in H$ since we can set $a=b=0$ and then $0+0=0$.

② Let $u = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, $v = \begin{bmatrix} a' & 0 \\ 0 & b' \end{bmatrix}$

$u+v = \begin{bmatrix} a+a' & 0 \\ 0 & b+b' \end{bmatrix}$ $(a+a') + (b+b') = (a+b) + (a'+b') = 0+0=0$ ✓

③ Let k be a scalar

$ku = k\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} ka & 0 \\ 0 & kb \end{bmatrix}$ $ka + kb = k(a+b) = k \cdot 0 = 0$ ✓

10. (8 points) Let $T : V \longrightarrow W$ be an one-to-one linear transformation.
Suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent subset of V.
Prove that $S' = \{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ is a linearly independent subset of W.

proof Suppose $c_1T(v_1) + c_2T(v_2) + c_3T(v_3) = 0_W$. (We must show $c_1 = c_2 = c_3 = 0$)

Since T is linear this can be written $T(c_1v_1 + c_2v_2 + c_3v_3) = 0_W$

Since T is one-to-one, $T(\vec{x}) = 0_W \Rightarrow \vec{x} = 0_V$

Thus $c_1v_1 + c_2v_2 + c_3v_3 = 0_V$

Now since $\{v_1, v_2, v_3\}$ is L.I. in V, $c_1 = c_2 = c_3 = 0$. $\boxtimes$