MATH 70 EXAM II SOLNS SPRING 2013

① T T T F F

② (a)

$$U = \begin{bmatrix} 1 & 5 & 0 & 3 & 1 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{array}{c} \begin{array}{ccccc} p & f & p & f & p \end{array} \\ \begin{bmatrix} 1 & 5 & 0 & 3 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$x_1 + 5x_2 + 3x_4 = 0$

$x_2 = x_2$

$x_3 + 2x_4 = 0$

$x_4 = x_4$

$x_5 = 0$

$x_1 = -5x_2 - 3x_4$

$x_2 = x_2$

$x_3 = -2x_4$

$x_4 = x_4$

$x_5 = 0$

Basis for Nul A $= \left\{ \begin{bmatrix} -5 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} \right\}$

(b) Basis for Col A $\left\{ \begin{bmatrix} 1 \\ -2 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 9 \\ 3 \end{bmatrix} \right\}$

(c) Basis for Row A

$\{(1,5,0,3,0), (0,0,1,2,0), (0,0,0,0,1)\}$

or

$\{(1,5,0,3,1), (0,0,1,2,-1), (0,0,0,0,1)\}$

(d) 3

(e) 3

(f) 2

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③ $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$

(a) $\ker T = \{ M \in M_{2\times 2} \mid T(M) = 0 \} = \left\{ \begin{bmatrix} a & 0 \\ c & d \end{bmatrix} : a, c, d \in \mathbb{R} \right\}$

(b) range of $T = \left\{ \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \mid b \in \mathbb{R} \right\} = \text{span}\left\{ \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \right\}$

(c) T is not 1-1 because $\ker T \neq 0$

(d) T is not onto since range of $T \neq M_{2\times 2}$. It is a 1-dim'l subspace of $M_{2\times 2}$.

(4) a) $[A]_B = \begin{bmatrix} 3 \\ -6 \\ 4 \\ 0 \end{bmatrix}$ $\quad [C]_B = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

b) $\{A, C\}$ is LI because neither vector is a multiple of the other

c) $M_{2\times 2}$ is isomorphic to $\mathbb{R}^4$ because it has dimension 4,

An isomorphism is $[\ ]_B : M_{2\times 2} \to \mathbb{R}^4$

(5) Many examples.

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ $\quad \det(A+B) = 0$ but $\det A + \det B$

$= 1+1 = 2$

(6) $\begin{vmatrix} 1 & 2 & 0 & 0 \\ 0 & -2 & 4 & 1 \\ 0 & 2 & -2 & 4 \\ 2 & 4 & 1 & 0 \end{vmatrix}_{-2R_1} = \begin{vmatrix} 1 & 2 & 0 & 0 \\ 0 & -2 & 4 & 1 \\ 0 & 2 & -2 & 4 \\ 0 & 0 & 1 & 0 \end{vmatrix}_{+R2} = \begin{vmatrix} 1 & 2 & 0 & 0 \\ 0 & -2 & 4 & 1 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 1 & 0 \end{vmatrix}$

$= 1 \begin{vmatrix} -2 & 4 & 1 \\ 0 & 2 & 5 \\ 0 & 1 & 0 \end{vmatrix} = 1\,(-1) \begin{vmatrix} -2 & 1 \\ 0 & 5 \end{vmatrix} = -(-10) = 10$

(7) Yes, no, no

(8) a) 4

b) $\frac{1}{4}$

c) $4^3 = 64$

d) $5^4 \cdot 4 = 2500$

(9) $W = \text{Span}\{u_1, u_2, u_3\}$ where $u_1, u_2, u_3 \in V$ a vector space

① $0_V \in W$ because $0 = 0 \cdot u_1 + 0 \cdot u_2 + 0 \cdot u_3$

② Let $v_1, v_2 \in W$, $v_1 = c_1u_1 + c_2u_2 + c_3u_3$, $v_2 = d_1u_1 + d_2u_2 + d_3u_3$

Then $v_1 + v_2 = (c_1u_1 + c_2u_2 + c_3u_3) + (d_1u_1 + d_2u_2 + d_3u_3)$

$= (c_1 + d_1)u_1 + (c_2 + d_2)u_2 + (c_3 + d_3)u_3 \in W$

so W is closed under $+$.

③ Let $v \in W$, c = scalar, $v = c_1u_1 + c_2u_2 + c_3u_3$

Then $cv = c(c_1u_1 + c_2u_2 + c_3u_3) = (cc_1)u_1 + (cc_2)u_2 + (cc_3)u_3 \in W$

So W is closed under scalar mult.

$\therefore$ W is a subspace of V.

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(10) (a) $1 \cdot p_1(t) - \frac{1}{2} p_2(t) + 1\, p_3(t)$

$= (1+t) - \frac{1}{2}(4) + (1-t) = 0$ ✓

(b) Since $\{p_1, p_2, p_3\}$ is LD, a basis for $\text{Span}\{p_1, p_2, p_3\} \neq \{p_1, p_2, p_3\}$.

There are 3 possibilities:

$\{p_1, p_2\}$ or $\{p_1, p_3\}$ or $\{p_2, p_3\}$

All of these work because each set is L.I and spans $\text{Span}\{p_1, p_2, p_3\}$

(11) $\underset{10\times 7}{A}$ $\underset{7\times 7}{B}$ $\dim(\mathrm{Nul}\,B)=2$ cols of A LI

(a) Prove $\dim \mathrm{Nul}(AB) = 2$

This proof will show $\mathrm{Nul}\,B = \mathrm{Nul}\,AB$ hence $\dim(\mathrm{Nul}\,B) = \dim(\mathrm{Nul}\,AB)$.

(i) Suppose $\vec{x} \in \mathrm{Nul}\,B$. Then $B\vec{x} = 0$

But $(AB)(\vec{x}) = A(B\vec{x}) = A0 = 0$ so $(AB)\vec{x} = 0$

ii) Suppose $\vec{x} \notin \mathrm{Nul}\,B$. Then $B\vec{x} \neq 0$.

$(AB)(\vec{x}) = A(\underset{\text{not } 0}{\underset{\uparrow}{B\vec{x}}}) \neq 0$ because the cols of A are LI so

$A\vec{v} = \vec{0}$ if and only if $\vec{v} = 0$

$\therefore$ $\mathrm{Nul}\,B = \mathrm{Nul}\,AB$ so $\dim(\mathrm{Nul}\,B) = \dim(\mathrm{Nul}\,AB)$

(b) $\dim(\mathrm{col}\,AB)$

$$\begin{aligned} \#\text{ of columns of } AB &= \dim(\mathrm{Nul}\,AB) + \dim(\mathrm{col}\,AB) \\ 7 &= 2 + \dim(\mathrm{col}\,AB) \end{aligned}$$

$\therefore$ $\dim(\mathrm{col}(AB)) = 5$