Math 70
Linear Algebra

TUFTS UNIVERSITY
Department of Mathematics

Apr 9, 2018
Sections 1,2,3,4
Exam 2

Instructions: No notes or books are allowed. All calculators, cell phones, or other electronic devices **must** be turned off and put away during the exam. Unless otherwise stated, you **must show all work** to receive full credit. *You are required to sign your exam. With your signature you are pledging that you have neither given nor received assistance on the exam. Students found violating this pledge will receive an F in the course.*

Problem	Point Value	Points
1	6	
2	6	
3	12	
4	12	
5	8	
6	7	
7	14	
8	11	
9	8	
10	16	
	100	

1. (6 pts) Find the inverse of $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 0 & 1 & 3 \end{bmatrix}$.

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array}\right] -R2 \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \begin{array}{l} -3R3 \\ -2R3 \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 1 & -3 \\ 0 & 1 & 0 & 3 & 0 & -2 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

↑ A^{-1}

check $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 0 & 1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 1 & -3 \\ 3 & 0 & -2 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ✓

2. (6 pts) T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$. Show that T is invertible:

$$T\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} -5x_1 + 9x_2 \\ 4x_1 - 7x_2 \end{pmatrix}$$

Method: Find the standard matrix A of T and show that A is invertible.

$$A = \left(T\begin{pmatrix} 1 \\ 0 \end{pmatrix} \; T\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} -5 & 9 \\ 4 & -7 \end{pmatrix}$$

$\det(A) = (-5)(-7) - (4)(9) = 35 - 36 = -1 \neq 0$

A is invertible $\Rightarrow$ T is invertible with $T^{-1}(x) = A^{-1}x$

3. (12 pts) **True/false** questions. For each of the statements below, decide whether it is true or false. Indicate your answer by shading the corresponding box. There will be no partial credit.

(a) (2 pts) If A is a 5×5 matrix and the equation $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$ in $\mathbb{R}^5$, it is possible that for some $\mathbf{b}$, the equation $A\mathbf{x} = \mathbf{b}$ has more than one solution.

T F

No - $x \mapsto Ax$ is onto and since A is 5×5,

$x \mapsto Ax$ is also 1-1

(b) (2 pts) It is possible for a 4×4 matrix to be invertible when its columns do not span $\mathbb{R}^4$.

T F

(c) (2 pts) A square matrix with two identical columns is invertible.

T F

(d) (2 pts) If $A^3 = 0$ then det $A = 0$.

T F

$\det A^3 = (\det(A))^3$

$\det 0 = 0$

(e) (2 pts) The pivot columns of a matrix are always linearly independent.

T F

(f) (2 pts) If A is $m \times n$ and rank $A = m$, then the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is onto.

T F

A has a pivot position in every row.

4. (12 pts) Suppose that a matrix A is row equivalent to the matrix B given below:

$$B = \begin{bmatrix} 2 & 0 & 3 & -1 \\ 0 & 4 & 1 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & 6 \end{bmatrix}$$

$$\left|\begin{matrix} 2 & 0 & 3 & -1 \\ 0 & 4 & 1 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & 6 \end{matrix}\right| \begin{matrix} \\ \\ \\ -2R3 \end{matrix} = \left|\begin{matrix} 2 & 0 & 3 & -1 \\ 0 & 4 & 1 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{matrix}\right| = 0$$

↑ row of zeros

(a) (3 pts) Compute the determinant of B.

0 (see above)

(b) (3 pts) Can you compute the determinant of A? If so, what is it? If not, why not?

yes - $\det(A) = 0$ $A \not\sim I$ so A is not invertible.

$$\left(A \sim \begin{pmatrix} 2 & 0 & 3 & -1 \\ 0 & 4 & 1 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right)$$

Suppose that a matrix C is row equivalent to the matrix D given below:

$$D = \begin{bmatrix} 2 & 0 & 3 & -1 \\ 0 & 4 & 1 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

(c) (3 pts) Compute the determinant of D.

Since D is triangular, $\det(D) = (2)(4)(1)(6) = 48$

(d) (3 pts) Can you compute the determinant of C? If so, what is it? If not, why not?

No - we do not know how many row interchanges occurred.

We only know $\det(A) = \pm 48$

5. (8 pts)

(a) (4 pts) Define what it means for a subset H to be a subspace of a vector space V.

1) $0_V \in H$

2) For all $u, v \in H$, $u+v \in H$

3) For all $u \in H$, scalars c, $cu \in H$.

(b) (4 pts) Let V be a vector space and let $\mathcal{B} = \{v_1, v_2, ..., v_p\}$ be an indexed set of vectors in V. Complete the following: $\mathcal{B}$ is called a basis for V if:

1) $\mathcal{B}$ is linearly independant

2) $\mathcal{B}$ spans V

6. (7 pts) Prove whether each of the following sets is a subspace. **If it does form a subspace, determine its dimension.**

   (a) (3 pts) $\{p(t) \in \mathbb{P}_3 : p(2) = 2\} \subseteq \mathbb{P}_3$

Not a subspace since $\theta_{\mathbb{P}^3} \notin$ of this set.

$\theta_{\mathbb{P}_3}(t) = 0$ for all t so $\theta_{\mathbb{P}_3}(2) = 0$, not 2.

   (b) (4 pts) $W = \left\{ \begin{bmatrix} 2t+2 \\ t+1 \\ 0 \\ -t-1 \end{bmatrix} : t \in \mathbb{R} \right\} \subseteq \mathbb{R}^4$

$W = \left\{ \begin{pmatrix} 2t \\ t \\ 0 \\ -t \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ 0 \\ -1 \end{pmatrix} : t \in \mathbb{R} \right\}$

$= \left\{ t\begin{pmatrix} 2 \\ 1 \\ 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ 0 \\ -1 \end{pmatrix} : t \in \mathbb{R} \right\}$

$= \left\{ s\begin{pmatrix} 2 \\ 1 \\ 0 \\ -1 \end{pmatrix} : s \in \mathbb{R} \right\} = \text{Span}\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ -1 \end{pmatrix} \right\}$

$\therefore$ W is a subspace of $\mathbb{R}^4$.

$\dim W = 1$

Alternate Solution to 6(b) Showing W is a subspace of $\mathbb{R}^4$

$$W = \left\{ \begin{pmatrix} 2t+2 \\ t+1 \\ 0 \\ -t-1 \end{pmatrix} : t \in \mathbb{R} \right\} \subseteq \mathbb{R}^4$$

① $0_{\mathbb{R}^4} \in W$ - set $t = -1$

② Let $u, v \in W$, $u = \begin{pmatrix} 2a+2 \\ a+1 \\ 0 \\ -a-1 \end{pmatrix}$, $v = \begin{pmatrix} 2b+2 \\ b+1 \\ 0 \\ -b-1 \end{pmatrix}$

$$u+v = \begin{pmatrix} 2a+2+2b+2 \\ a+1+b+1 \\ 0 \\ -a-1-b-1 \end{pmatrix} = \begin{pmatrix} 2(a+b)+4 \\ a+b+2 \\ 0 \\ -a-b-2 \end{pmatrix} \overset{\star}{=} \begin{pmatrix} 2(a+b+1)+2 \\ (a+b+1)+1 \\ 0 \\ (-a-b-1)-1 \end{pmatrix} \in W$$

let $t = a+b+1$

③ Let $u \in W$, c be a scalar
as above

$$cu = c\begin{pmatrix} 2a+2 \\ a+1 \\ 0 \\ -a-1 \end{pmatrix} = \begin{pmatrix} 2ca+2c \\ ca+c \\ 0 \\ -ca-c \end{pmatrix} \overset{?}{=} \begin{pmatrix} 2t+2 \\ t+1 \\ 0 \\ -t-1 \end{pmatrix} = \begin{pmatrix} 2(ca+c-1)+2 \\ (ca+c-1)+1 \\ 0 \\ -(ca+ca-1)-1 \end{pmatrix}$$

$2ca+2c = 2t+2$

$ca+c = t+1$

$ca+c-1 = t$

check

$ca+c \overset{?}{=} (ca+c-1)+1$ ✓ ($t+1$)

check

$-ca-c \overset{?}{=} -t-1$

$-ca-c \overset{?}{=} -(ca+c-1)-1$

$= -ca-c+1$ ✓

7. (14 pts) Let $T : \mathbb{P}_2 \to \mathbb{R}^2$ be the transformation defined by

$$p(t) \mapsto \begin{bmatrix} p(0) \\ p''(0) \end{bmatrix}$$

where p'' denotes the second derivative of p.

(a) (8 pts) Prove that T is a linear transformation.

1) Show $T(p+q) = T(p) + T(q)$ where $p, q \in \mathbb{P}_2$

$$T((p+q)(t)) = \begin{bmatrix} (p+q)(0) \\ (p+q)''(0) \end{bmatrix} = \begin{bmatrix} p(0) + q(0) \\ p''(0) + q''(0) \end{bmatrix} = \begin{bmatrix} p(0) \\ p''(0) \end{bmatrix} + \begin{bmatrix} q(0) \\ q''(0) \end{bmatrix}$$

$$= T(p) + T(q) \checkmark$$

2) Show $T(cp) = cT(p)$ where $p \in \mathbb{P}_2$, c is a scalar

$$T((cp)(t)) = \begin{bmatrix} (cp)(0) \\ (cp)''(0) \end{bmatrix} = \begin{bmatrix} c \cdot p(0) \\ c \cdot p''(0) \end{bmatrix} = c \begin{bmatrix} p(0) \\ p''(0) \end{bmatrix} = c \cdot T(p) \checkmark$$

(b) (6 pts) Find a basis for the kernel of T.

Let $p(t) = a + bt + ct^2 \in \mathbb{P}_2$. Then $p(0) = a$, $p'(t) = b + 2ct$

$p''(t) = 2c$

So $T(p(t)) = \begin{bmatrix} a \\ 2c \end{bmatrix}$

$T(p(t)) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff a = 0$ and $2c = 0 \rightarrow c = 0$

$\therefore p(t) \in \ker T \iff p(t) = bt$

One basis for $\ker T$ is $\mathcal{B} = \{t\}$

8. (11 pts) Consider the matrix

$$A = \begin{bmatrix} 2 & 0 & 4 & 0 & 3 \\ 1 & 1 & 2 & 2 & 0 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & -1 & 0 & -2 & -1 \end{bmatrix} \text{ and its row equivalent matrix } B = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) (3 pts) Find a basis for Col A and determine its dimension.

$$\mathcal{B}_{\text{Col } A} = \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\} \qquad \dim \text{Col } A = 3$$

(b) (3 pts) Find a basis for Nul A and determine its dimension.

$$\left[\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 + 2x_3 = 0$, $x_2 + 2x_4 = 0$, $x_3 = x_3$, $x_4 = x_4$, $x_5 = 0$

$x_1 = -2x_3$, $x_2 = -2x_4$, $x_3 = x_3$, $x_4 = x_4$, $x_5 = 0$

$$\vec{x} = x_3 \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ -2 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Basis $\mathcal{B} = \left\{ \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

$\dim \text{Nul } A = 2$

(c) (3 pts) Find a basis for Row A and determine its dimension.

$$\mathcal{B}_{\text{Row } A} = \{ (1,0,2,0,0), (0,1,0,2,0), (0,0,0,0,1) \}$$

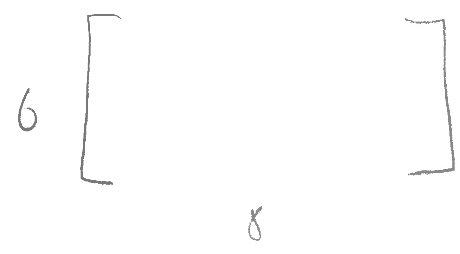
$\dim \text{Row } A = 3$

(d) (2 pts) How many pivot columns are in a row echelon form of A^T? Explain.

3 = # of pivot columns of A

9. (8 pts) Suppose a nonhomogeneous system of six equations of eight unknowns has a solution for all possible constants on the right sides of the equations. Is it possible to find two nonzero solutions of the associated homogeneous system that are not multiples of each other? Explain.

$A \quad x = b$



Given: $Ax=b$ has a solution for all $b \in \mathbb{R}^6$ → A has pivot positions in all 6 rows → A has 6 pivot columns and

$Ax=0$ has exactly 2 free variables.

Since $Ax=0$ has exactly 2 free variables, dim Nul A = 2.

Thus Nul A has a basis of size 2 and these 2 vectors satisfy the condition of being 2 non-0 solutions that are LI (not multiples)

<u>YES</u>

10. (16 pts) Let V be the set of all 2×2 matrices.

Let H be the subspace of V with a basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ and consider the ordered set,

$$S = \{u, v, w\} = \left\{ \begin{bmatrix} -6 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 4 & 5 \\ 0 & 10 \end{bmatrix}, \begin{bmatrix} -2 & 10 \\ 0 & 20 \end{bmatrix} \right\}.$$

(a) (4 pts) Write $[u]_\mathcal{B}$, $[v]_\mathcal{B}$, and $[w]_\mathcal{B}$ in $\mathcal{B}$-coordinate vectors.

$[u]_B = \begin{pmatrix} -6 \\ 0 \\ 0 \end{pmatrix}$ $[v]_B = \begin{pmatrix} 4 \\ 5 \\ 10 \end{pmatrix}$ $[w]_B = \begin{pmatrix} -2 \\ 10 \\ 20 \end{pmatrix}$

(b) (8 pts) Is $S = \{u, v, w\} = \left\{ \begin{bmatrix} -6 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 4 & 5 \\ 0 & 10 \end{bmatrix}, \begin{bmatrix} -2 & 10 \\ 0 & 20 \end{bmatrix} \right\}$ linearly independent? Explain.

Consider $S' = \{[u]_B, [v]_B, [w]_B\} = \left\{ \begin{pmatrix} -6 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 10 \end{pmatrix}, \begin{pmatrix} 2 \\ -10 \\ 20 \end{pmatrix} \right\}$

Since the coordinate mapping is an isomorphism, S is LI $\Leftrightarrow$ S' is LI.

Let $A = \begin{pmatrix} -6 & 4 & 2 \\ 0 & 5 & -10 \\ 0 & 10 & 20 \end{pmatrix} \underset{-2R_2}{\sim} \begin{pmatrix} -6 & 4 & 2 \\ 0 & 5 & -10 \\ 0 & 0 & 0 \end{pmatrix}$ The columns of A are LD since $A \not\sim I$.

$\therefore$ Since the columns of A were the vectors in S', S' is LD
$\Rightarrow$ S is LD.

(Problem 10 continues on the next page)

(Problem 10 continues)

(c) (4 pts) Find a basis for the subspace spanned by S.

The 3 vectors in S are LD but no 2 are multiples of each other and none are $\mathbb{C}$. Choose any 2 for a basis —

e.g. $\mathcal{B} = \left\{ \begin{pmatrix} -6 & 0 \\ 0 & 6 \end{pmatrix}, \begin{pmatrix} 4 & 5 \\ 0 & 1 \end{pmatrix} \right\}$