KEY

Math 70
Linear Algebra

TUFTS UNIVERSITY
Department of Mathematics

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Exam II

Instructions: No notes or books are allowed. All calculators, cell phones, or other electronic devices **must** be turned off and put away during the exam. Unless otherwise stated, you **must show all work** to receive full credit. *You are required to sign your exam. With your signature you are pledging that you have neither given nor received assistance on the exam. Students found violating this pledge will receive an F in the course.*

Problem	Point Value	Points
1	6	
2	4	
3	14	
4	8	
5	6	
6	10	
7	10	
8	10	
9	10	
10	8	
11	6	
12	8	
	100	

1. (6 pts) Let $A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Is A invertible? If so, find A^{-1}.

Since A is (upper) triangular, det(A) = product of diagonal entries = (1)(1)(1) = 1

Since det(A) = 1 ≠ 0 A is invertible.

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} -2R_3 \\ -R_3 \end{array} \sim \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & -2 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] -3R_2 \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -3 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

Claim: $A^{-1} = \begin{bmatrix} 1 & -3 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

Check $AA^{-1} = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -3 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ✓

2. (4 pts) Let V be a vector space and let $B = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ be an indexed set of vectors in V. Complete the following: B is a basis for V if

① B is linearly independent.

② B spans V.

3. (14 pts) Let V and W be vector spaces.

(a) Let $T : V \to W$ be a transformation. Give the conditions from the definition for T to be a *linear* transformation.

① $T(u+v) = T(u) + T(v)$ for all $u, v \in V$

② $T(cu) = c\,T(u)$ for all $u \in V$ and all scalars c.

(b) Define the transformation $T : \mathbb{P}_2 \to \mathbb{R}_2$ by $T(p(t)) = \begin{bmatrix} p(2) \\ p(0) \end{bmatrix}$.

(i) Use your definition from part (a) to prove T is a linear transformation.

Let $p(t)$ and $q(t)$ be any two polynomials in $\mathbb{P}_2$. Let c be any scalar.

① $T((p+q)(t)) = \begin{bmatrix} (p+q)(2) \\ (p+q)(0) \end{bmatrix} = \begin{bmatrix} p(2) + q(2) \\ p(0) + q(0) \end{bmatrix} = \begin{bmatrix} p(2) \\ p(0) \end{bmatrix} + \begin{bmatrix} q(2) \\ q(0) \end{bmatrix} = T(p(t)) + T(q(t))$ ✓

② $T((cp)(t)) = \begin{bmatrix} (cp)(2) \\ (cp)(0) \end{bmatrix} = \begin{bmatrix} c\,p(2) \\ c\,p(0) \end{bmatrix} = c \begin{bmatrix} p(2) \\ p(0) \end{bmatrix} = c\,T(p(t))$ ✓

The map $T : \mathbb{P}_2 \to \mathbb{R}_2$ is given by $T(p(t)) = \begin{bmatrix} p(2) \\ p(0) \end{bmatrix}$.

(ii) Find a basis for the kernel of T.

$$\text{Ker } T = \left\{ p(t) \in \mathbb{P}_2 \;\middle|\; T(p(t)) = 0_{\mathbb{R}^2} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

Let $p(t) = a + bt + ct^2$ be a polynomial in $\mathbb{P}^2$

Then $p(t) \in \text{Ker } T \iff T(p(t)) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\iff \begin{pmatrix} p(2) \\ p(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\iff \begin{pmatrix} a + (b)2 + (c)4 \\ a \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Solve: $\left. \begin{array}{l} a + 2b + 4c = 0 \\ a = 0 \end{array} \right\}$ $\quad 2b + 4c = 0$, $\quad 2b = -4c$ or $b = -2c$

$\left[\text{or } c = -\frac{1}{2} b\right]$

$\therefore$ $p(t) \in \text{Ker } T \iff p(t) = 0 - 2ct + ct^2$ where c is any scalar.

$\text{Ker } T = \text{Span}\{-2t + t^2\}$

Since $p(t) = -2t + t^2$ is not the zero polynomial, it is linearly independent.

Basis for $\text{Ker } T = \{-2t + t^2\}$

4. (8 pts) Consider the polynomials $\mathbf{p}_1(t) = 1 + t$, $\mathbf{p}_2(t) = t + t^2$, $\mathbf{p}_3(t) = 3 + 3t + t^2$. Determine whether or not $S = \{\mathbf{p}_1(t), \mathbf{p}_2(t), \mathbf{p}_3(t)\}$ is linearly independent in $\mathbb{P}_2$. Explain your procedure.

Let $\mathcal{B} = \{1, t, t^2\}$. Then $\mathbb{P}_2 \cong \mathbb{R}^3$ via the coordinate

is isomorphic to

Mapping $p(t) \to [p(t)]_{\mathcal{B}}$. Let $S' = \{[p_1(t)]_{\mathcal{B}}, [p_2(t)]_{\mathcal{B}}, [p_3(t)]_{\mathcal{B}}\}$.

Then S is linearly independent in $\mathbb{P}_2$ if and only if S' is linearly independent in $\mathbb{R}^3$.

$$S' = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix} \right\}$$

Let $A = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix}$. We need to determine if the columns of A are linearly independent.

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 1 & 1 & 3 \\ 0 & 1 & 1 \end{pmatrix} -R_1 \sim \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 3 \\ 0 & 1 & 1 \end{pmatrix} -R_2 \sim \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since A has a pivot position in every column, $Ax = 0$ has only the solution $x = 0$ so, yes the columns of A are linearly independent $\Rightarrow S'$ is linearly independent $\Rightarrow S$ is linearly independent. $\boxtimes$

5. (6 pts) Let V be the vector space of all continuous, real-valued functions. Let H be the subspace of V with a basis $\mathcal{B} = \{e^t, \cos(t), t^2, 1\}$.

(a) The dimension of H is $\underline{4}$. = size of basis $\mathcal{B}$.

(b) Find $[4 + 2\cos(t) + e^t + 3t^2]_{\mathcal{B}}$.

Since $4 + 2\cos(t) + e^t + 3t^2 = 1 \cdot e^t + 2 \cdot \cos(t) + 3 \cdot t^2 + 4 \cdot 1$,

$$[4 + 2\cos t + e^t + 3t^2]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \in \mathbb{R}^4$$

6. (10 pts) Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ where $\mathbf{b}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

(a) Find the vector $\mathbf{x}$ whose coordinate vector relative to $\mathcal{B}$ is $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$.

Let $P_{\mathcal{B}} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$. then $P_{\mathcal{B}} [x]_{\mathcal{B}} = x$.

$$\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} -2 \\ 5 \end{pmatrix} = \begin{pmatrix} -9 \\ 7 \end{pmatrix} = x.$$

(b) Suppose $\mathbf{x} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$. Find the coordinates of $\mathbf{x}$ relative to $\mathcal{B}$ - i.e. find $[\mathbf{x}]_{\mathcal{B}}$.

option 1 Since $P_{\mathcal{B}} [x]_{\mathcal{B}} = x$, $[x]_{\mathcal{B}} = P_{\mathcal{B}}^{-1} x$ $\quad P_{\mathcal{B}}^{-1} = \frac{1}{1}\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$

$[x]_{\mathcal{B}} = P_{\mathcal{B}}^{-1} x = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$

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option 2 Solve for $[x]_{\mathcal{B}}$: $P_{\mathcal{B}} [x]_{\mathcal{B}} = x$

$$\left(\begin{array}{cc|c} 2 & -1 & 4 \\ -1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} -1 & 1 & 0 \\ 2 & -1 & 4 \end{array}\right) \xrightarrow{-R_1} \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 2 & -1 & 4 \end{array}\right) \xrightarrow{-2R_1} \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 1 & 4 \end{array}\right) \xrightarrow{+R_1} \left(\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 4 \end{array}\right)$$

$[x]_{\mathcal{B}} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$

7. (10 pts) Let $A = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 3 & 4 & 5 \\ 5 & 4 & 0 & 3 \\ 2 & 0 & 0 & 1 \end{pmatrix}$. Compute $\det(A)$ and use the result to decide whether or not A is invertible.

$$|A| = \begin{vmatrix} 1 & 0 & 0 & 2 \\ 0 & 3 & 4 & 5 \\ 5 & 4 & 0 & 3 \\ 2 & 0 & 0 & 1 \end{vmatrix} -2R_1 = \begin{vmatrix} 1 & 0 & 0 & 2 \\ 0 & 3 & 4 & 5 \\ 5 & 4 & 0 & 3 \\ 0 & 0 & 0 & -3 \end{vmatrix} = -3 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 3 & 4 \\ 5 & 4 & 0 \end{vmatrix}$$

$$\begin{matrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{matrix}$$

$$= -3(1) \begin{vmatrix} 3 & 4 \\ 4 & 0 \end{vmatrix} = -3(1)(-16) = 48$$

Since $|A| = 48 \neq 0$, A is invertible.

8. (10 pts) Let $A = \begin{bmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 2 & -4 \\ 1 & 0 & 1 & 2 & -1 \end{bmatrix}$.

Find a basis for each of the following spaces (show your work) :

(a) Nul(A)

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 2 & -4 \\ 1 & 0 & 1 & 2 & -1 \end{pmatrix}_{-R_1} \sim \begin{pmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 2 & -4 \\ 1 & 0 & 1 & 2 & -4 \end{pmatrix}_{-R_2} \sim \begin{array}{c} \begin{matrix} p & f & p & f & f \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 2 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

$x_1 + 3x_5 = 0$

$x_2 = x_2$

$x_3 + 2x_4 - 4x_5 = 0$

$x_4 = x_4$

$x_5 = x_5$

$x_1 = -3x_5$

$x_2 = x_2$

$x_3 = -2x_4 + 4x_5$

$x_4 = x_4$

$x_5 = x_5$

$$x = x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -3 \\ 0 \\ 4 \\ 0 \\ 1 \end{pmatrix}$$

$$\text{Basis for Nul } A = \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 4 \\ 0 \\ 1 \end{pmatrix} \right\}$$

(b) Col(A)

Basis for Col A consists of the pivot columns of $\underline{A}$

$$\text{Basis for Col } A = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

9. (10 pts)

(a) Let H be a subset of a vector space V. Give the conditions from the definition for H to be a subspace.

① $0_V \in H$

② $u+v \in H$ for all $u, v \in H$.

③ $cu \in H$ for all $u \in H$ and all scalars c.

(b) Use the definition from part (a) to show that $H = \{a + bt^2 \in \mathbb{P}_2 \mid a + b = 0\}$ is a subspace of $\mathbb{P}_2$.

① The zero polynomial of $\mathbb{P}_2$ is $p(t) = 0$ for all t. This can also be written $p(t) = 0 + 0 \cdot t + 0 \cdot t^2$ so since $0+0=0$, $p(t)=0$ is in H.

② Suppose $p(t) = a + bt^2$ and $q(t) = a' + b't^2$ are both in H.
Then $a+b=0$ and $a'+b'=0$

$$(p+q)(t) = p(t) + q(t) = (a+bt^2) + (a'+b't^2)$$
$$= (a+a') + (b+b')t^2$$
$$(a+a') + (b+b') = (a+b) + (a'+b') = 0+0=0$$

③ Suppose $p(t) = a+bt^2 \in H$. (so $a+b=0$) let c be an scalar.

$$(cp)(t) = c(p(t)) = c(a+bt^2) = ca + cbt^2$$

$ca + cb = c(a+b) = c(0) = 0$. Thus $(cp)(t) \in H$. $\boxtimes$

10. (8 points) Suppose a nonhomogeneous system of 10 linear equations in 12 unknowns is *inconsistent*. Is it possible to find 3 solutions of the associated homogeneous system that are linearly independent? Explain.

Let $Ax=b$ represent the non-homogeneous system where A is 10×12 and $b \in \mathbb{R}^{10}$.

Since $Ax=b$ is inconsistent, A does not have a pivot position in every row, and so A has at most 9 pivot positions.

This means $Ax=0$, the associated homogeneous system, has at least $12-9=3$ free variables.

Hence $\dim(\text{Nul } A) \geq 3$. Any basis for Nul A will contain at least 3 linearly independent vectors. Each of these vectors is a solution to $Ax=0$.

The answer to this question is yes.

11. (6 pts) Suppose that the columns of a $n \times n$ matrix B are linearly dependent. Show that the columns of AB are also linearly dependent for any $n \times n$ matrix A.

(Hint: the columns of a $n \times n$ matrix C are linearly independent if and only if $\det(C) \neq 0$.)

Let A and B be $n \times n$ matrices with the columns of B linearly dependent. Since the columns of B are linearly dependent, $\det(B) = 0$.

Now, $\det(AB) = \det(A)\det(B) = \det(A) \cdot 0 = 0$.

$\therefore$ The columns of AB are linearly dependent. $\boxtimes$

12. (8 pts) Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ be a basis for a vector space V. Prove that $\mathcal{B}' = \{\mathbf{b}_1 + \mathbf{b}_2, \mathbf{b}_1 - \mathbf{b}_2\}$ is also a basis for V. Explain your procedure and show all work.

I. Since V has a basis of 2 vectors, $\dim V = 2$. By the Basis Theorem since the size of $\mathcal{B}'$ is $2 = \dim(V)$, we can prove that $\mathcal{B}'$ is a basis for V by either just showing that $\mathcal{B}'$ is linearly independent or just showing that $\mathcal{B}'$ spans V. (we get the other property "for free" by the Basis Theorem)

Option 1: Show $\mathcal{B}'$ is linearly independent.

Suppose $x_1(b_1+b_2) + x_2(b_1-b_2) = 0_V$ → We need to show that $x_1=0$ and $x_2=0$

(distribute) $x_1 b_1 + x_1 b_2 + x_2 b_1 - x_2 b_2 = 0_V$

(regroup) $(x_1+x_2)b_1 + (x_1-x_2)b_2 = 0_V$

We know that since $\mathcal{B}$ is a basis + hence linearly independent so if $\square b_1 + \diamond b_2 = 0_V$ then $\square = 0$ and $\diamond = 0$

Since $\mathcal{B}$ is a basis and hence linearly independent we know

$x_1 + x_2 = 0$
$x_1 - x_2 = 0$

Solving:

$$\begin{array}{r} x_1 + x_2 = 0 \\ x_1 - x_2 = 0 \\ \hline 2x_1 \quad = 0 \\ x_1 = 0 \end{array}$$

→ but $x_1 + x_2 = 0$ → $0 + x_2 = 0$ → $x_2 = 0$

$\therefore \mathcal{B}' = \{b_1+b_2, b_1-b_2\}$ is linearly independent → $\mathcal{B}'$ is a basis for V.

12-2

option 2. Show that $\mathcal{B}'$ spans V.

Let $v \in V$ be any vector. We must show $\exists$ scalars x_1 and x_2 with

$\boxed{1}$ $v = x_1(b_1 + b_2) + x_2(b_1 - b_2)$.

(ie we must show that we can write v as a linear combination of the vectors in $\mathcal{B}'$.)

Since $v \in V$ and $\mathcal{B} = \{b_1, b_2\}$ is a basis for V, we know $\exists$ unique fixed scalars c_1 and c_2 such that

$\boxed{2}$ $v = c_1 b_1 + c_2 b_2$.

[Method: Solve for x_1 and x_2 in terms of the known values c_1 and c_2]

From $\boxed{1}$: $v = x_1 b_1 + x_1 b_2 + x_2 b_1 - x_2 b_2$

$v = (x_1 + x_2) b_1 + (x_1 - x_2) b_2$

From $\boxed{2}$ $v = c_1 b_1 + c_2 b_2$

$\therefore$

$$\begin{aligned} x_1 + x_2 &= c_1 \\ x_1 - x_2 &= c_2 \\ \hline 2x_1 &= c_1 + c_2 \end{aligned}$$

$$\boxed{x_1 = \frac{c_1 + c_2}{2}}$$

Since $x_1 + x_2 = c_1$

$\frac{c_1 + c_2}{2} + x_2 = c_1$

$x_2 = c_1 - \frac{c_1 + c_2}{2}$

$x_2 = \frac{2c_1}{2} - \frac{c_1 + c_2}{2} =$

$x_2 = \frac{2c_1 - (c_1 + c_2)}{2} = \boxed{\frac{c_1 - c_2}{2}}$

$\therefore v \in V \to v \in \text{Span}\{b_1 + b_2, b_1 - b_2\}$

$\to \mathcal{B}'$ spans $V \to \mathcal{B}'$ is a basis for V.

12-3 OPTION 3

II. Given a basis $\mathcal{B} = \{b_1, b_2\}$ for V and a set $\mathcal{B}' = \{b_1+b_2, b_1-b_2\}$,

we can show that $\mathcal{B}'$ is a basis for V (which has dimension 2)

by showing that $S = \{ [b_1+b_2]_{\mathcal{B}}, [b_1-b_2]_{\mathcal{B}} \}$ is a basis for $\mathbb{R}^2$. ($V \cong \mathbb{R}^2$)

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}. \quad \text{Let } A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The cols of A are linearly independent ($\exists$ a pivot in every column of A) and

span $\mathbb{R}^2$ ($\exists$ a pivot in every row of A), $\therefore$ S is a basis for $\mathbb{R}^2$.

Since $V \cong \mathbb{R}^2$, $\mathcal{B}'$ is a basis for V.