KEY (corrected)

Math 70
Linear Algebra

TUFTS UNIVERSITY
Department of Mathematics
Exam 2

Nov 5, 2012
All sections

Instructions: No notes or books are allowed. All calculators, cell phones, or other electronic devices **must** be turned off and put away during the exam. Unless otherwise stated, you **must show all work** to receive full credit. *You are required to sign your exam. With your signature you are pledging that you have neither given nor received assistance on the exam. Students found violating this pledge will receive an F in the course.*

Throughout this exam:

- $\mathbb{C}[0,1]$ represents the vector space of all continuous real-valued functions defined on the closed interval $[0,1]$,
- $\mathbb{P}_n$ represents the vector space of polynomials of degree less than or equal to n, and
- $M_{m\times n}$ is the vector space of matrices of size $m \times n$.

We will also denote by $|A|$ the determinant of A.

Problem	Point Value	Points
1	10	
2	9	
3	10	
4	10	
5	6	
6	9	
7	18	
8	8	
9	4	
10	8	
11	8	
	100	

1. (10 pts) **True/false questions.** For each of the statements below, decide whether it is true or false. Indicate your answer by shading the corresponding box. There will be no partial credit.

   (a) Let $\mathbb{P}_2$ be the vector space of all polynomials of degree less than or equal to 2. Let
   $$S = \{1+t, t^2\}$$
   be a *set* of vectors in $\mathbb{P}_2$. Then, the polynomial $p(t) = 1$ does not belong to Span S. T F

   T - $\nexists$ c_1, c_2 with $1 = c_1(1+t) + c_2 t^2$

   (b) The subset W of $\mathbb{R}^2$ given by $\left\{ \begin{bmatrix} 3t+1 \\ -t \end{bmatrix} : t \in \mathbb{R} \right\}$ is a subspace of $\mathbb{R}^2$. T F

   $0 \notin W$

   (c) If A is a square matrix such that $|A| = 0$, then $A\vec{x} = \vec{0}$ has a nontrivial solution. T F

   cols of A L.D.

   (d) For any $A, B \in M_{n \times n}$, $|A+B| = |A| + |B|$. T F

   $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

   (e) Let $A \in M_{m \times n}$. Then $\dim \mathrm{Col} A + \dim \mathrm{Nul} A = n$. T F

2. (9 pts)

(a) Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ be a set of vectors in a vector space V. What are the two conditions that $\mathcal{B}$ must satisfy for it to be a basis of V?

① $\mathcal{B}$ must be a linearly independent set.

② $V = \text{Span}\{\vec{b}_1 \ldots \vec{b}_n\}$

(b) Give the *definition* of linear independence of a set of vectors $S = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p\}$ in a vector space V.

dfn S is l.i. if the only solution to

$$c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_n\vec{v}_n = 0$$

is $c_1 = c_2 = \cdots = c_n = 0$.

3. (10 pts) Determine whether or not each of the following sets S is linearly independent in the given vector space V. Indicate your answer by shading the corresponding box. No explanation needed. No partial credit.

(a) $V = \mathbb{R}^3$ and $S = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} \right\}$. $\boxed{\text{Yes}}$ $\boxed{\text{No}}$

$u \quad v \quad w$

$(v = 2u)$

(b) $V = \mathbb{P}_1$ and $S = \{3 + t, 6 + t\}$. $\boxed{\text{Yes}}$ $\boxed{\text{No}}$

$3+t \neq \theta$

$6+t \neq c(3+t)$

(c) $V = \mathbb{C}[0, 1]$ and $S = \{e^x, e^{-x}, 2e^x + 4e^{-x}\}$. $\boxed{\text{Yes}}$ $\boxed{\text{No}}$

$u \quad v \quad w$

$w = 2u + 4v$

4. (10 pts)

(a) The following transformation is linear. (You do **not** need to prove it.)

$$T: \quad \mathbb{P}_2 \longrightarrow \mathbb{R}^2$$
$$p(t) \longmapsto \begin{bmatrix} p(0) \\ p(-1) \end{bmatrix}$$

(i) Find a set of vector(s) in $\mathbb{P}_2$ that span $\ker(T)$.

$\ker T = \left\{ p(t) \in \mathbb{P}_2 \mid \begin{bmatrix} p(0) \\ p(-1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$

$p(t) = a + bt + ct^2$ Need $p(0) = a = 0 \rightarrow a = 0$

$p(0) = a$ Need $-b + c = 0 \rightarrow b = c$

$p(-1) = -b + c$

So $p(t) = bt + bt^2$

$\ker T = \text{Span}\{t + t^2\}$ so $S = \{t + t^2\}$

(ii) Is the linear transformation T one-to-one? Briefly justify your answer.

Since $\ker T \neq \mathcal{O}$, T is <u>not</u> 1-1.

(or $T(\mathcal{O}) = \mathcal{O}_{\mathbb{R}^2}$

$T(t + t^2) = \mathcal{O}_{\mathbb{R}^2}$ so T is not 1-1)

(b) Let k be a real number and T be the transformation given by

$$T: \quad \mathbb{P}_2 \longrightarrow \mathbb{R}^3$$
$$p(t) \longmapsto \begin{bmatrix} k \\ p(0) \\ p(-1) \end{bmatrix}$$

For which value(s) of k is T a linear transformation? Briefly justify your answer.

Since T is linear $\rightarrow T(\mathcal{O}_{\mathbb{P}_2}) = \mathcal{O}_{\mathbb{R}^3}$

we would need $k = 0$.

5. (6 points) In each of the following you are given a vector space V and an ordered basis for V. You will be asked to find some coordinate vectors. No partial credit.

(a) $V = \mathbb{P}_3$ and $\mathcal{B} = \{1, t, t^2, t^3\}$. Find $[2 - t + 5t^2]_{\mathcal{B}}$.

$$[2 - t + 5t^2]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \\ 5 \\ 0 \end{bmatrix}$$

(b) $V = \mathbb{P}_3$ and $\mathcal{B} = \{t^3, t^2, t, 1\}$. Find $[5t - 11t^3]_{\mathcal{B}}$.

$$[5t - 11t^3]_{\mathcal{B}} = \begin{bmatrix} -11 \\ 0 \\ 5 \\ 0 \end{bmatrix}$$

1 3 2 4

(c) $V = M_{2\times 2}$ and $\mathcal{B} = \{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)\}$. Find $[\left(\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right)]_{\mathcal{B}}$.

$$\left[\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\right]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 4 \end{bmatrix}$$

6. (9 pts) Compute the following determinants:

(a) $|D^{101}|$, where $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & \sqrt{2} & -1 \end{bmatrix}$

$$|D^{101}| = |D|^{101} = (-1)^{101} = -1$$

↑

D is triangular $\rightarrow |D| = (1)(1)(-1)$

(b) $\begin{vmatrix} 1 & -1 & 0 & 2 \\ -1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 1 \\ 3 & 1 & 5 & -1 \end{vmatrix} \begin{matrix} \\ +R_1 \\ -2R_1 \\ -3R_1 \end{matrix}$ $= \begin{vmatrix} 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 4 & 5 & -7 \end{vmatrix} = - \begin{vmatrix} 1 & -1 & 0 & 2 \\ 0 & 4 & 5 & -7 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 2 \end{vmatrix}$

$-3 \quad 3 \quad 0 \quad -6$

$$= -(1)(4)(1)(2) = -8$$

7. (18 pts) Assume that the matrix A is row equivalent to the matrix B.

$$A = \begin{bmatrix} 1 & 2 & 4 & 8 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & 0 & -2 & -8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Do the columns of A span $\mathbb{R}^4$? Why or why not?

No - the columns of A are not even in $\mathbb{R}^4$ - they are in $\mathbb{R}^3$

(b) Do the columns of A span $\mathbb{R}^3$? Why or why not?

No - A has only 2 pivot columns so dim (col A) = 2 - can't span $\mathbb{R}^3$.

(c) Find a basis for Col A.

$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \\ 9 \end{bmatrix} \right\}$

(d) What is the rank of A?

2 = dim Col A

Assume that the matrix A is row equivalent to the matrix B.

$$A = \begin{bmatrix} 1 & 2 & 4 & 8 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & 0 & -2 & -8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(e) Find a basis for Nul A. Write your answer in set notation.

$$\left[\begin{array}{cccc|c} 1 & 2 & 4 & 8 & 0 \\ 0 & 0 & -2 & -8 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & 4 & 8 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \overset{-4R_2}{\sim} \left[\begin{array}{cccc|c} 1 & 2 & 0 & -8 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 + 2x_2 - 8x_4 = 0$

$x_2 = x_2$

$x_3 + 4x_4 = 0$

$x_4 = x_4$

$x_1 = -2x_2 + 8x_4$

$x_2 = x_2$

$x_3 = -4x_4$

$x_4 = x_4$

$$\vec{x} = s\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 8 \\ 0 \\ -4 \\ 1 \end{bmatrix}$$

$$\mathcal{B} = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ 0 \\ -4 \\ 1 \end{bmatrix} \right\}$$

(f) Check that the vector(s) found in (e) really are in Nul A.

$$\begin{bmatrix} 1 & 2 & 4 & 8 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2+2 \\ -4+4 \\ -6+6 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \checkmark$$

$$\begin{bmatrix} 1 & 2 & 4 & 8 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \end{bmatrix}\begin{bmatrix} 8 \\ 0 \\ -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 8-16+8 \\ 16-24+8 \\ 24-36+12 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \checkmark$$

8. (8 pts) Let $\mathcal{B} = \left\{ \begin{bmatrix} 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \end{bmatrix} \right\}$ be a basis for $\mathbb{R}^2$.

(a) Find $\mathcal{P}_\mathcal{B}$.

$$\mathcal{P}_B = \begin{bmatrix} 4 & 3 \\ 7 & 5 \end{bmatrix}$$

(b) Given $\vec{x} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, find $[\vec{x}]_\mathcal{B}$.

$$\mathcal{P}_B [x]_B = \vec{x} \quad \text{So} \quad [x]_B = \mathcal{P}_B^{-1} \vec{x}$$

$$\mathcal{P}_B^{-1} = -1 \begin{bmatrix} 5 & -3 \\ -7 & 4 \end{bmatrix} = \begin{bmatrix} -5 & 3 \\ 7 & -4 \end{bmatrix}$$

$$[x]_B = \begin{bmatrix} -5 & 3 \\ 7 & -4 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ 6 \end{bmatrix}$$

(c) Given $[\vec{x}]_\mathcal{B} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, find $\vec{x}$.

$$\mathcal{P}_B [x]_B = \vec{x}$$

$$\begin{bmatrix} 4 & 3 \\ 7 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

9. (4 pts)
(a) If A is a 7×4 matrix what is the smallest possible dimension for Nul A?

dim Nul A + dim col A = 4
min 0   max 4

0

(b) If A is a 3×10 matrix what is the smallest possible dimension for Nul A?

dim Nul A + dim col A = 10
min 7   max 3

7

10. (8 pts) Show that the set $S = \{1, 2t, -2+4t^2, -12t+8t^3\}$ forms a basis for $\mathbb{P}_3$.

We will use coordinate vectors in $\mathbb{R}^4$. If those vectors are L.I. in $\mathbb{R}^4$, then the polynomials are L.I. in $\mathbb{P}_3$.

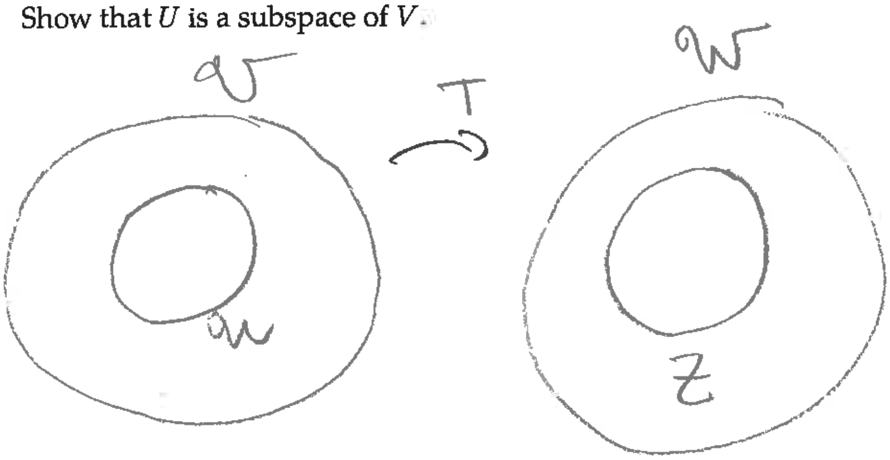
Let $\mathcal{B} = \{1, t, t^2, t^3\}$

$$A = \begin{bmatrix} [1]_{\mathcal{B}} & [2t]_{\mathcal{B}} & [-2+4t^2]_{\mathcal{B}} & [-12t+8t^3]_{\mathcal{B}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 2 & 0 & -12 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

We can see that every column of A is a pivot column $\Rightarrow$ cols of A L.I. in $\mathbb{R}^4$ $\Rightarrow$ polynomials are L.I. in $\mathbb{P}_3$.

Since dim $\mathbb{P}_3 = 4$ and S is a set of 4 L.I. vectors, S is a basis for $\mathbb{P}_3$.

11. (8 pts) Let V and W be vector spaces, and let $T : V \to W$ be a linear transformation. Given a subspace Z of W, let U be the set of all $\mathbf{x} \in V$ such that $T(\mathbf{x}) \in Z$. Show that U is a subspace of V.



① Show $0_V \in U$.

Since T is linear $T(0_V) = 0_W$. Since Z is a subspace of W, $0_W \in Z$. Thus $T(0_V) \in Z$ and $0_V \in U$.

② Show U is closed under $+$:

Let $u, v \in U$. So $T(u), T(v) \in Z$.

Since Z is a subspace $T(u) + T(v) \in Z$. But T is linear so $T(u) + T(v) = T(u+v) \in Z$.

Hence $u+v \in U$.

③ Show U is closed under scalar mult.

Let $u \in U$ and c be a scalar.

$u \in U \Rightarrow T(u) \in Z$, Since Z is a subspace, $cT(u) \in Z$. But T is linear so $cT(u) = T(cu) \Rightarrow$

$cu \in U$. $\boxtimes$