Math 46 | Tufts University
Department of Mathematics
Exam 2 | November 10, 2008

1. (10 points) True/False. Please write the word "true" or "false".

   (i) If A and B are two $n \times n$ matrices such that $\det(AB) = 4$, then A and B are both invertible.

   (ii) If H is a subset of a vector space V such that $\vec{0} \in H$, then H is a subspace of V.

   (iii) If A is an $n \times n$ matrix and $\mathbf{v} \in \mathbb{R}^n$ is a nonzero vector, then the set of solutions of the matrix equation $A\mathbf{x} = \mathbf{v}$ is a subspace of $\mathbb{R}^n$.

   (iv) If a set of vectors $\{v_1, \ldots, v_m\}$ in a vector space V spans V, then $\dim V > m$.

   (v) The vector space $\mathbb{P}$ of all polynomials in one variable is infinite-dimensional.

2. (10 points) Set $\mathcal{B} = \{v_1, v_2\}$, where $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

   (i) Using only determinants, show that $\mathcal{B}$ is a basis of $\mathbb{R}^2$ (do *not* use row operations).

   (ii) If $[v]_{\mathcal{B}} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$, find v.

   (iii) Find the change-of-coordinates matrix $P_{\mathcal{B}}$ (recall that $P_{\mathcal{B}}$ is the matrix such that $v = P_{\mathcal{B}}\,[v]_{\mathcal{B}}$, for all $v \in \mathbb{R}^2$).

3. (10 points) Let V be a finite-dimensional vector space with basis $\mathcal{B} = \{v_1, \ldots, v_n\}$.

(i) If $v = c_1v_1 + c_2v_2 + \cdots + c_nv_n$, then what is $[v]_{\mathcal{B}}$?

(ii) The coordinate mapping $v \mapsto [v]_{\mathcal{B}}$ is a linear map from V to what vector space?

(iii) Show that the coordinate mapping $v \mapsto [v]_{\mathcal{B}}$ is one-to-one by choosing two arbitrary vectors u and v and showing that if $[u]_{\mathcal{B}} = [v]_{\mathcal{B}}$ then $u = v$.

4. (6 points) Determine whether each of the following is a subspace of $\mathbb{P}_2$. You do not need to show your work; no work will be graded.

(i) $\{a + bt + ct^2 : b = 0\}$.

(ii) $\{a + bt + ct^2 : b = c\}$.

(iii) $\{a + bt + ct^2 : a = 3\}$.

5. (14 points) Determinants.

(i) Find the determinant of $A = \begin{pmatrix} 3 & 1 & 2 \\ 4 & 0 & 1 \\ 9 & 6 & 1 \end{pmatrix}$ using cofactor expansion.

(ii) Suppose that A is a square matrix with $\det A = 4$. Find the determinant of B, where the matrix B is obtained from A by each of the following operations:

(a) Switching two rows of A.

(b) Multiplying a row of A by $-\frac{1}{3}$.

(c) Adding twice the first row of A to the third row of A.

(d) Multiplying A by itself 3 times (ie, $B = A^3$).

(e) Inverting A.

6. (12 points) Linear independence.

(i) Define what it means for a collection of vectors $\{v_1, \ldots, v_m\}$ in a vector space V to be linearly independent.

(ii) Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be a basis of a vector space V and let $v \in V$ be any vector. Show that the set $\{v_1, \ldots, v_n, v\}$ is linearly dependent by finding a dependence relation.

(iii) Let $\mathcal{B} = \{1, t, t^2\}$ be the standard basis of $\mathbb{P}_2$. Use the coordinate mapping associated to $\mathcal{B}$ to determine whether or not the set

$$\mathcal{S} = \{1 + 2t + 3t^2,\ 4 + 5t + 6t^2,\ 2 + t\}$$

is linearly independent.

7. (12 points) Let $A = \begin{pmatrix} 1 & -3 & -2 & 12 & -4 \\ 0 & 7 & 7 & -12 & 11 \\ -4 & -7 & -11 & -13 & -9 \\ -1 & 0 & -1 & -7 & -1 \end{pmatrix}$. Then A is row equivalent to the matrix

$$B = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(you do not have to verify this). Find bases for

(i) row A

(ii) col A

(iii) nul A

(iv) nul B

8. (14 points) Suppose that A is a 4×11 matrix. Find the following:

(i) The maximum value of rank A.

(ii) The minimum value of rank A.

(iii) The maximum value of dim(nul A).

(iv) The minimum value of dim(nul A).

(v) Row A is a subspace of ________ .

(vi) Col A is a subspace of ________ .

(vii) Nul A is a subspace of ________ .

9. (12 points) Let V and W be vector spaces, and let $T : V \to W$ be a linear map.

(i) Define the kernel of T.

(ii) Define the range of T.

(iii) Prove that the kernel of T is a subspace of V. **WARNING:** Do *NOT* be tempted to use matrices.