Math 70 | **TUFTS UNIVERSITY** | **March 30, 2020**
Linear Algebra | **Department of Mathematics** | **All sections 1,2,3,4,5**
Exam 2

Instructions:

- This is an open book and notes, timed exam. You are not allowed to collaborate with any other person. All calculators, cell phones, or other electronic devices **must** be turned off and put away during the exam.
- There are three possible ways to complete the exam:
  1. Print out the exam and write directly on your copy
  2. Write on blank paper (each numbered problem on a separate sheet – clearly label each problem – include a signed honor pledge at the end)
  3. Write on the exam using a pdf editing program (e.g., on a tablet)
- Once you finish your solutions, scan them, and submit **a single pdf** file to Canvas. Make sure you leave yourself about 10 minutes at the end to scan and upload your test!
- Unless otherwise stated, you **must show all work** to receive full credit.

You are required to sign your exam. With your signature you are pledging that you have neither given nor received assistance from any person on the exam. Students found violating this pledge will receive an F in the course.

Problem	Point Value	Points
1	12	
2	12	
3	8	
4	8	
5	8	
6	10	
7	8	
8	12	
9	11	
10	11	
	100	

1. (12 points) Indicate by shading the appropriate box whether each statement is true or false. *For this problem you do not need to give reasons.*

Let A **be a** 6×4 **matrix**.

(a) The dimension of the column space of A could be any natural number less than or equal to 4. T F

(b) The dimension of the row space of A could be any natural number less than or equal to 6. T F

(c) If A is the standard matrix of a linear transformation T, then T could be an isomorphism. T F

Now A**,** B**,** C**, and** U **are arbitrary matrices**.

(d) If A and U are row-equivalent matrices, and the determinant of A is 0, then the determinant of U is 0. T F

(e) If A has linearly dependent columns, and B has linearly independent columns, then AB has linearly independent columns. T F

(f) If A, B, and C are $n \times n$ matrices and $AC = BC$ then $A = B$. T F

2. (12 points) Suppose B is the reduced echelon form for the matrix A.

$$A = \begin{bmatrix} 1 & 3 & 1 & -8 & 0 \\ 1 & 3 & -2 & -11 & 0 \\ -2 & -6 & -2 & 16 & 0 \\ -4 & -12 & -4 & 32 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 3 & 0 & -9 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Find a basis for $\mathrm{Row}\ A$.

(b) Find a basis for $\mathrm{Nul}\ A$.

(c) Find a basis for $\mathrm{Col}\ A$.

(d) Give an example of one dependence relation among the columns of A.

3. (8 points) Let $\mathcal{B} = \{\vec{b_1}, \vec{b_2}\}$, where $\vec{b_1} = \begin{bmatrix} -1 \\ -4 \end{bmatrix}$ and $\vec{b_2} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

   (a) Find the vector $\vec{x}$ whose coordinate vector relative to $\mathcal{B}$ is given by: $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

   (b) Suppose $\vec{x} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$. Find the coordinates of $\vec{x}$ relative to $\mathcal{B}$ - i.e. find $[\vec{x}]_{\mathcal{B}}$.

4. (8 points) For what $h, k \in \mathbb{R}$ is the following matrix A invertible?

$$A = \begin{bmatrix} 1 & 6 & 2 & -1 \\ 2 & h & 9 & 7 \\ -2 & -12 & k & 6 \\ -1 & -6 & -2 & 3 \end{bmatrix}$$

5. Let $\mathbb{P}_2$ be the vector space consisting of polynomials with real coefficients with degree at most 2. Recall that every vector in $\mathbb{P}_2$ can be written as $a_2t^2 + a_1t + a_0$, where a_2, a_1, a_0 are real numbers. Consider the following subset of $\mathbb{P}_2$:

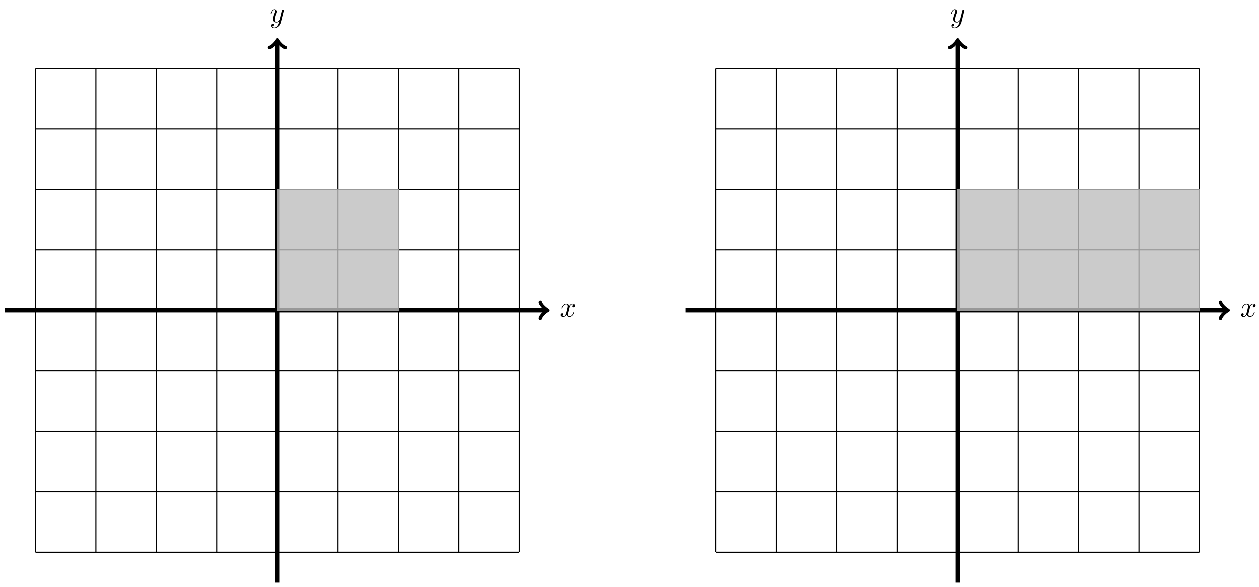
$$H = \{a_2t^2 + a_1t + a_0 : a_0 + a_1 = a_2\}$$

(a) (3 pts) Is H a subspace of $\mathbb{P}_2$?

(b) (5 pts) Prove your answer. If you are showing that H is a subspace, verify the conditions in the definition of subspace. If you are showing that H is not a subspace, give an *explicit* example of vectors in H which do not satisfy one of these conditions.

6. (10 points)

(a) Let T be a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ that maps the gray square in the first grid to the gray rectangle in the second grid. Let A be the standard matrix, so that $T(\vec{v}) = A\vec{v}$.

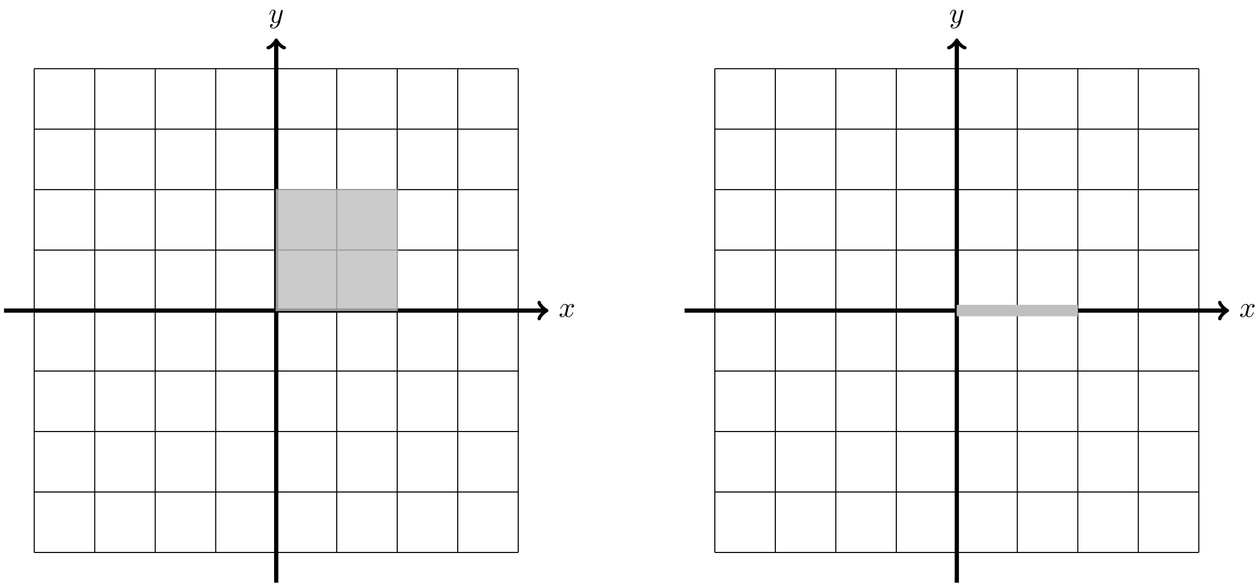


Hint for this problem: You can find a specific transformation that takes the first gray square on your left to the gray rectangle on your right. All such linear transformations will have a standard matrix A with the same $|\det A|$.

(i) What is the absolute value of the determinant of A, $|\det A|$?

(ii) Is T an invertible transformation? Briefly explain your answer.

(b) Let S be a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ that maps the gray square in the first grid to the gray line segment in the second grid. Let B be the standard matrix of S, so that $S(\vec{v}) = B\vec{v}$.



(i) What is the absolute value of the determinant of B, $|\det B|$?

(ii) Is S an invertible transformation? Briefly explain your answer.

7. (8 points) Let $M_{2\times 2}$ be the vector space consisting of 2×2 matrices with real entries.
   (a) Give a basis for $M_{2\times 2}$. You do not have to prove that this is a basis.

   (b) What is the dimension of $M_{2\times 2}$?

   (c) What is the dimension of the subspace H of $M_{2\times 2}$, where $H = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : 2a - b + d = 0 \right\}$?
   Briefly explain your answer.

8. Let $T : \mathbb{P}_3 \to \mathbb{P}_1$ be the linear transformation from the vector space of polynomials with degree at most 3 to the vector space of polynomials with degree at most 1 given by

$$T(a_3t^3 + a_2t^2 + a_1t + a_0) = (a_3 - a_1)t + (a_2 + 2a_0)$$

(a) (3 pts) Write down a system of two linear equations (in 4 unknowns) whose solutions describe the kernel of T.

(b) (3 pts) Write down a basis for the kernel of T. You do not have to prove that your answer is a basis.

(c) (3 pts) What is the dimension of the kernel of T? Briefly explain your answer.

(d) (3 pts) What is the dimension of the range of T? Briefly explain your answer.

9. Let $T : V \to W$ be a linear transformation between vector spaces V and W. Prove that, if $\{T(v_1), T(v_2), T(v_3)\}$ is a linearly dependent set, and $T : V \to W$ is one-to-one, then $\{v_1, v_2, v_3\}$ is a linearly dependent set of vectors in V.
   (a) (2 pts) As the first part of the proof, give the definition for a linear transformation $T : V \to W$ to be one-to-one.

   (b) (2 pts) As the next part of the proof, give the definition for a set of vectors to be linearly dependent.

   (c) (7 pts) Now finish the proof.

10. Let V be a vector space and $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ a set of vectors in V. Prove that $\text{Span}\{\vec{v}_1, \vec{v}_2 + \vec{v}_3\}$ is a subspace of V.
    (a) (2 pts) As the first part of the proof, write the definition of a subspace of a vector space V.

    (b) (2 pts) As the second part of the proof, write the definition of the span of a set of vectors.

    (c) (7 points) Now finish the proof.

Math 70 **Exam 2** **March 30, 2020**

Name: ______________________________

Please circle your section

Section 1, Eunice Kim, TTh 10:30–11:45

Section 2, Genevieve Walsh, TTh 1:30–2:45

Section 3, Yanghui Liu, MW 9:00–10:15

Section 4, Curtis Heberle, TTh 12:00–1:15

Section 5, Nathan Fisher, MW 9:00–10:15

I pledge that I have neither given nor received assistance on this exam.

Signature ______________________________