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| **Math 70** | **TUFTS UNIVERSITY** | **Nov 5, 2012** |
| **Linear Algebra** | **Department of Mathematics** | **All sections** |
| | **Exam 2** | |

**Instructions:** No notes or books are allowed. All calculators, cell phones, or other electronic devices **must** be turned off and put away during the exam. Unless otherwise stated, you **must show all work** to receive full credit. *You are required to sign your exam. With your signature you are pledging that you have neither given nor received assistance on the exam. Students found violating this pledge will receive an F in the course.*

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Throughout this exam:

- $\mathbb{C}[0, 1]$ represents the vector space of all continuous real-valued functions defined on the closed interval $[0, 1]$,
- $\mathbb{P}_n$ represents the vector space of polynomials of degree less than or equal to $n$, and
- $M_{m\times n}$ is the vector space of matrices of size $m \times n$.

We will also denote by $|A|$ the determinant of $A$.

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| Problem | Point Value | Points |
|---|---|---|
| 1 | 10 | |
| 2 | 9 | |
| 3 | 10 | |
| 4 | 10 | |
| 5 | 6 | |
| 6 | 9 | |
| 7 | 18 | |
| 8 | 8 | |
| 9 | 4 | |
| 10 | 8 | |
| 11 | 8 | |
| | 100 | |

1. (10 pts) **True/false questions.** For each of the statements below, decide whether it is true or false. Indicate your answer by shading the corresponding box. There will be no partial credit.

   (a) Let $\mathbb{P}_2$ be the vector space of all polynomials of degree less than or equal to 2. Let
   $$S = \{1 + t, t^2\}$$
   be a *set* of vectors in $\mathbb{P}_2$. Then, the polynomial $p(t) = 1$ does not belong to Span $S$. T F

   (b) The subset $W$ of $\mathbb{R}^2$ given by $\left\{ \begin{bmatrix} 3t+1 \\ -t \end{bmatrix} : t \in \mathbb{R} \right\}$ is a subspace of $\mathbb{R}^2$. T F

   (c) If $A$ is a square matrix such that $|A| = 0$, then $A\vec{x} = \vec{0}$ has a nontrivial solution. T F

   (d) For any $A, B \in M_{n\times n}$, $|A + B| = |A| + |B|$. T F

   (e) Let $A \in M_{m\times n}$. Then $\dim \text{Col}A + \dim \text{Nul}A = n$. T F

2. (9 pts)

(a) Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ be a set of vectors in a vector space $V$. What are the two conditions that $\mathcal{B}$ must satisfy for it to be a basis of $V$?

(b) Give the *definition* of linear independence of a set of vectors $S = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p\}$ in a vector space $V$.

3. (10 pts) Determine whether or not each of the following sets $S$ is linearly independent in the given vector space $V$. Indicate your answer by shading the corresponding box. No explanation needed. No partial credit.

(a) $V = \mathbb{R}^3$ and $S = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} \right\}$. Yes No

(b) $V = \mathbb{P}_1$ and $S = \{3 + t, 6 + t\}$. Yes No

(c) $V = \mathbb{C}[0, 1]$ and $S = \{e^x, e^{-x}, 2e^x + 4e^{-x}\}$. Yes No

4. (10 pts)

(a) The following transformation is linear. (You do **not** need to prove it.)

$$\begin{array}{rccc} T: & \mathbb{P}_2 & \longrightarrow & \mathbb{R}^2 \\ & p(t) & \longmapsto & \begin{bmatrix} p(0) \\ p(-1) \end{bmatrix} \end{array}$$

(i) Find a set of vector(s) in $\mathbb{P}_2$ that span $\ker(T)$.

(ii) Is the linear transformation $T$ one-to-one? Briefly justify your answer.

(b) Let $k$ be a real number and $T$ be the transformation given by

$$\begin{array}{rccc} T: & \mathbb{P}_2 & \longrightarrow & \mathbb{R}^3 \\ & p(t) & \longmapsto & \begin{bmatrix} k \\ p(0) \\ p(-1) \end{bmatrix} \end{array}$$

For which value(s) of $k$ is $T$ a linear transformation? Briefly justify your answer.

5. (6 points) In each of the following you are given a vector space $V$ and an ordered basis for $V$. You will be asked to find some coordinate vectors. No partial credit.

   (a) $V = \mathbb{P}_3$ and $\mathcal{B} = \{1, t, t^2, t^3\}$. Find $[2 - t + 5t^2]_{\mathcal{B}}$.

   (b) $V = \mathbb{P}_3$ and $\mathcal{B} = \{t^3, t^2, t, 1\}$. Find $[5t - 11t^3]_{\mathcal{B}}$.

   (c) $V = M_{2\times 2}$ and $\mathcal{B} = \{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)\}$. Find $[\left(\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right)]_{\mathcal{B}}$.

6. (9 pts) Compute the following determinants:

(a) $|D^{101}|$, where $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & \sqrt{2} & -1 \end{bmatrix}$

(b) $\begin{vmatrix} 1 & -1 & 0 & 2 \\ -1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 1 \\ 3 & 1 & 5 & -1 \end{vmatrix}$

7. (18 pts) Assume that the matrix $A$ is row equivalent to the matrix $B$.

$$A = \begin{bmatrix} 1 & 2 & 4 & 8 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & 0 & -2 & -8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Do the columns of $A$ span $\mathbb{R}^4$? Why or why not?

(b) Do the columns of $A$ span $\mathbb{R}^3$? Why or why not?

(c) Find a basis for Col $A$.

(d) What is the rank of $A$?

Assume that the matrix $A$ is row equivalent to the matrix $B$.

$$A = \begin{bmatrix} 1 & 2 & 4 & 8 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & 0 & -2 & -8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(e) Find a basis for Nul $A$. Write your answer in set notation.

(f) Check that the vector(s) found in (e) really are in Nul $A$.

8. (8 pts) Let $\mathcal{B} = \left\{ \begin{bmatrix} 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \end{bmatrix} \right\}$ be a basis for $\mathbb{R}^2$.

(a) Find $\mathcal{P}_{\mathcal{B}}$.

(b) Given $\vec{x} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, find $[\vec{x}]_{\mathcal{B}}$.

(c) Given $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, find $\vec{x}$.

9. (4 pts)

(a) If $A$ is a $7 \times 4$ matrix what is the smallest possible dimension for Nul $A$?

(b) If $A$ is a $3 \times 10$ matrix what is the smallest possible dimension for Nul $A$?

10. (8 pts) Show that the set $S = \{1, 2t, -2 + 4t^2, -12t + 8t^3\}$ forms a basis for $\mathbb{P}_3$.

11. (8 pts) Let $V$ and $W$ be vector spaces, and let $T : V \to W$ be a linear transformation. Given a subspace $Z$ of $W$, let $U$ be the set of all $\mathbf{x} \in V$ such that $T(\mathbf{x}) \in Z$.
Show that $U$ is a subspace of $V$.

Math 70 Exam 2 November 5, 2012

Name ______________________________

Instructor ______________________________

I pledge that I have neither given nor received assistance on this exam.

Signature ______________________________