KEY

Math 70
Linear Algebra

TUFTS UNIVERSITY
Department of Mathematics

Feb 26, 2018
Sections 2,3,4
Exam 1

Instructions: No notes or books are allowed. All calculators, cell phones, or other electronic devices **must** be turned off and put away during the exam. Unless otherwise stated, you **must show all work** to receive full credit. *You are required to sign your exam. With your signature you are pledging that you have neither given nor received assistance on the exam. Students found violating this pledge will receive an F in the course.*

Problem	Point Value	Points
1	20	
2	10	
3	10	
4	8	
5	10	
6	14	
7	12	
8	8	
9	8	
	100	

1. (20 points) Consider the following system of equations:

$$x_1 - 2x_2 + 2x_3 + 10x_4 = 3$$
$$x_1 - 2x_2 + 5x_3 + 25x_4 = 9$$

(a) Write the system as a a vector equation.

$$x_1\begin{pmatrix}1\\1\end{pmatrix} + x_2\begin{pmatrix}-2\\-2\end{pmatrix} + x_3\begin{pmatrix}2\\5\end{pmatrix} + x_4\begin{pmatrix}10\\25\end{pmatrix} = \begin{pmatrix}3\\9\end{pmatrix}$$

(b) Write the system as a matrix equation $A\mathbf{x} = \mathbf{b}$.

$$\begin{pmatrix}1 & -2 & 2 & 10\\1 & -2 & 5 & 25\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = \begin{pmatrix}3\\9\end{pmatrix}$$

(c) Solve the system and write your solution in parametric vector form.

$$\left(\begin{array}{cccc|c}1 & -2 & 2 & 10 & 3\\1 & -2 & 5 & 25 & 9\end{array}\right)\begin{matrix}\\-R_1\end{matrix} \sim \left(\begin{array}{cccc|c}1 & -2 & 2 & 10 & 3\\0 & 0 & 3 & 15 & 6\end{array}\right)\begin{matrix}\\(\tfrac{1}{3})\end{matrix} \sim \left(\begin{array}{cccc|c}1 & -2 & 2 & 10 & 3\\0 & 0 & 1 & 5 & 2\end{array}\right)\begin{matrix}-2R_2\\ \end{matrix} \sim \begin{matrix}p\,f\ \ p\,f\\ \left(\begin{array}{cccc|c}1 & -2 & 0 & 0 & -1\\0 & 0 & 1 & 5 & 2\end{array}\right)\end{matrix}$$

$x_1 - 2x_2 = -1$

$x_2 = x_2$

$x_3 + 5x_4 = 2$

$x_4 = x_4$

$x_1 = -1 + 2x_2$

$x_2 = x_2$

$x_3 = 2 - 5x_4$

$x_4 = x_4$

$$\vec{x} = \begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = \begin{pmatrix}-1\\0\\2\\0\end{pmatrix} + x_2\begin{pmatrix}2\\1\\0\\0\end{pmatrix} + x_4\begin{pmatrix}0\\0\\-5\\1\end{pmatrix}$$

x_2, x_4 scalars

or $\boxed{\vec{x} = \begin{pmatrix}-1\\0\\2\\0\end{pmatrix} + s\begin{pmatrix}2\\1\\0\\0\end{pmatrix} + t\begin{pmatrix}0\\0\\-5\\1\end{pmatrix} \quad s, t \text{ scalars}}$

2. (10 pts) **True/false questions.** For each of the statements below, decide whether it is true or false. Indicate your answer by shading the corresponding box. There will be no partial credit.

(a) The map $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(\mathbf{v}) = \mathbf{0}_{\mathbb{R}^m}$ for all $\mathbf{v}$ in $\mathbb{R}^n$ is linear. T F

(b) Let A be an $m \times n$ matrix.
The range of the map $\mathbf{x} \mapsto A\mathbf{x}$ is the span of the columns of A. T F

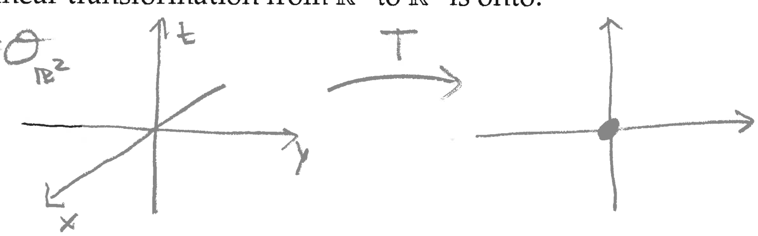
(c) Every set of four vectors in $\mathbb{R}^3$ is linearly dependent. T F

$$\underbrace{\begin{pmatrix} x & x & x & x \\ x & x & x & x \\ x & x & x & x \end{pmatrix}}_{A}$$

$A\mathbf{x} = \mathbf{0}$ must have free variables $\Rightarrow$ cols of A are L.D.

(d) Every linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$ is onto. T F

e.g. $T(x) = \mathbf{0}_{\mathbb{R}^2}$



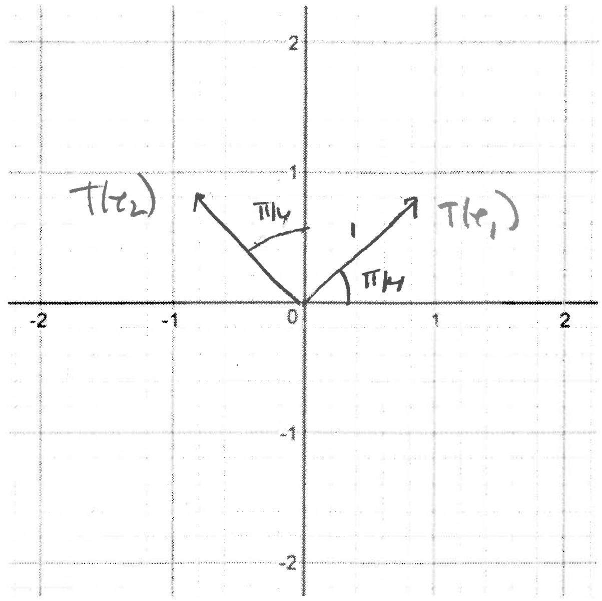
(e) If a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent, then one of the vectors is a scalar multiple of one of the others. T F

$$S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

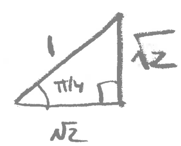
S is LD but no vector is a multiple of any other.

3. (10 points) Plot the images of $\mathbf{e}_1$ and $\mathbf{e}_2$ under the following linear transformations and then write down the standard matrix for each transformation.

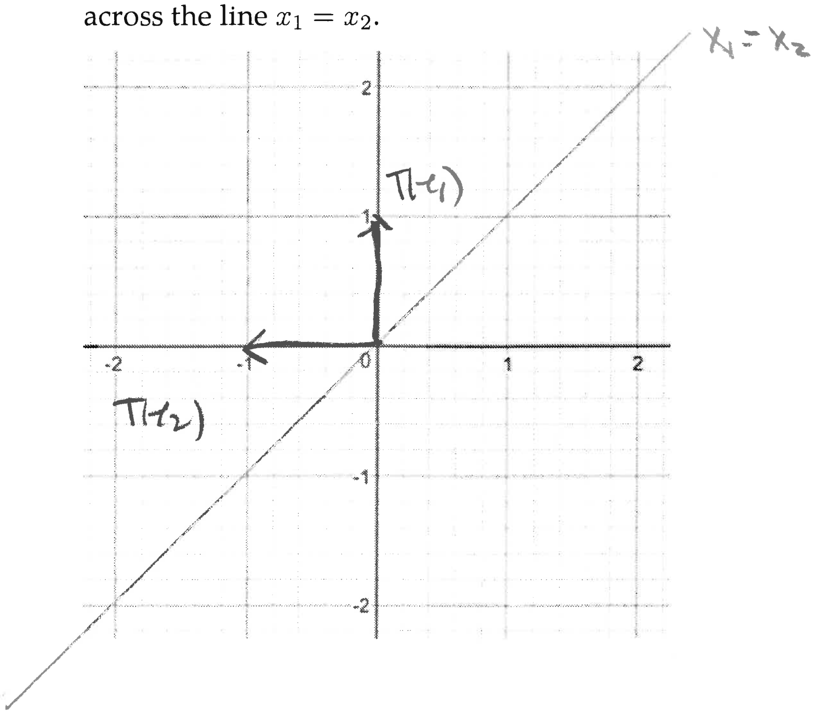
(a) $T : \mathbb{R}^2 \to \mathbb{R}^2$ rotates points (about the origin) $\pi/4$ radians.



$$\begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}$$

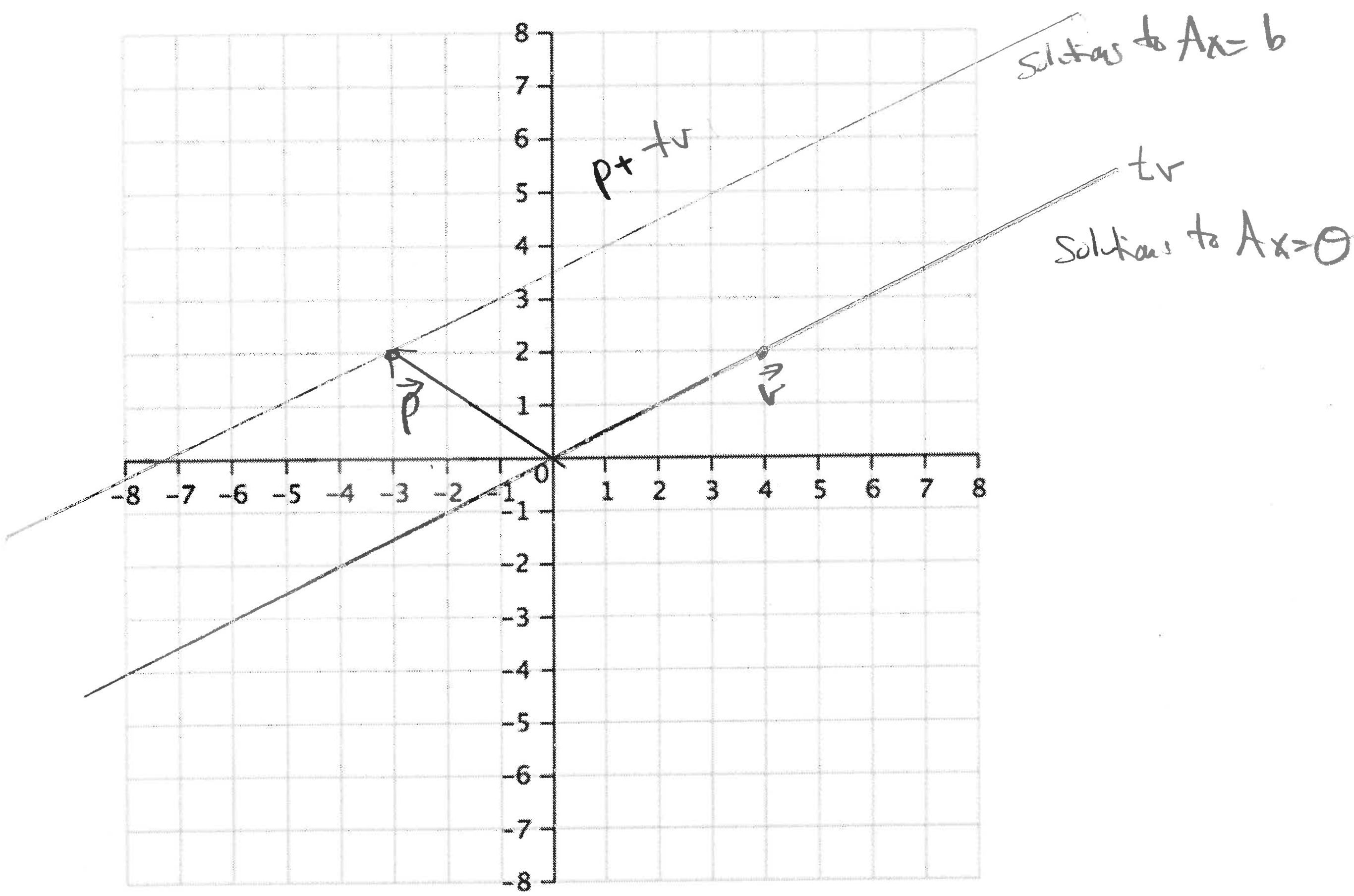


(b) $T : \mathbb{R}^2 \to \mathbb{R}^2$ first reflects points through the horizontal x_1-axis and then reflects points across the line $x_1 = x_2$.



$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

4. (8 points) Let A be a 2×2 matrix. Suppose the solution set to the homogeneous equation $A\mathbf{x} = \mathbf{0}$ is $\{t\mathbf{v} \mid \text{t is any scalar}\}$ where $\mathbf{v} = (4, 2)$. Suppose that $\mathbf{p} = (-3, 2)$ is a particular solution to the nonhomogeneous equation $A\mathbf{x} = \mathbf{b}$ for the same matrix A and for some nonzero vector $\mathbf{b}$. Sketch and clearly label the solution sets to $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{x} = \mathbf{b}$ on the grid below.



5. (10 points) 3×4

(a) Write down a matrix A in row echelon form that represents a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ that is onto but not one-to-one. Briefly explain why your matrix is correct.

$$\begin{array}{cc} & \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ A = & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

A has a pivot position in every row so $Ax=b$ has at least one solution for all $b \in \mathbb{R}^3 \Rightarrow T$ onto.

But x_4 is a free variable so $Ax=0$ has an infinite # of solutions so T is not 1-1.

(Many answers)
e.g.

$$A = \begin{pmatrix} 2 & 3 & 2 & 3 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 4 & 8 \end{pmatrix}$$

5×3

(b) Write down a matrix B in row echelon form that represents a linear transformation $S : \mathbb{R}^3 \to \mathbb{R}^5$ that is one-to-one but not onto. Briefly explain why your matrix is correct.

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

B has a pivot position in every column so $Bx=0$ has no free variables $\rightarrow$ $Bx=0 \Leftrightarrow x=0$ so T is 1-1.

Many answers

e.g.

$$B = \begin{pmatrix} 3 & 7 & 9 \\ 0 & 2 & 6 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

But B does not have a pivot position in row $\rightarrow$ there will be some $b \in \mathbb{R}^5$ such that $Bx=b$ has no solution $\Rightarrow$ T is not onto.

6. (14 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $T(x, y) = (3x, x - 2y, 0)$.

(a) Write down the *definition* of what it means to say that $T : \mathbb{R}^2 \to \mathbb{R}^3$ is **linear**.

$T : \mathbb{R}^2 \to \mathbb{R}^3$ is linear if for all vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^2$ and all scalars c,

① $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$

② $T(c\vec{u}) = cT(\vec{u})$

(b) Use your definition to show that T is linear. Show all steps.

Let $\vec{u} = (x,y)$, $\vec{v} = (x', y') \in \mathbb{R}^2$. Let c be a scalar.

① $T(\vec{u}+\vec{v}) = T((x,y) + (x',y'))$

$= T(x+x', y+y')$

$= (3(x+x'), (x+x') - 2(y+y'), 0)$

$= (3x+3x', x-2y+x'-2y', 0)$

$= (3x, x-2y, 0) + (3x', x'-2y', 0)$

$= T(\vec{u}) + T(\vec{v})$ $\boxtimes$

② $T(c\vec{u}) = T(c(x,y)) = T(cx, cy)$

$= (3cx, cx - 2cy, 0)$

$= (c(3x), c(x-2y), 0)$

$= c(3x, x-2y, 0)$

$= cT(\vec{u})$ $\boxtimes$

7. (12 points)

(a) Write the augmented matrix of the following linear system and row reduce it to echelon form.

$$x_1 + 5x_2 = k$$
$$4x_1 + hx_2 = 12$$

$$\left[\begin{array}{cc|c} 1 & 5 & k \\ 4 & h & 12 \end{array}\right] \begin{array}{c} \\ -4R_1 \end{array} \sim \left[\begin{array}{cc|c} 1 & 5 & k \\ 0 & h-20 & 12-4k \end{array}\right]$$

(b) Find all value(s) of h and k so the linear system above has

(i) no solutions

$h - 20 = 0$ $\quad h = 20$

$12 - 4k \neq 0$ $\quad 12 \neq 4k$ or $k \neq 3$

(ii) only one solution

$h - 20 \neq 0$ $\quad h \neq 20$

$12 - 4k =$ any # $\quad k =$ any #

(iii) infinitely many solutions

$h - 20 = 0$ $\quad h = 20$

$12 - 4k = 0$ $\quad k = 3$

8. (8 points) Let A be an $m \times 4$ matrix with columns $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$, i.e. $A = [\,\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3\ \mathbf{a}_4]$.

(a) Suppose $\mathbf{x} = \begin{bmatrix} 3 \\ -2 \\ -2 \\ 1 \end{bmatrix}$ is a solution to the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

Write down a dependence relation among the columns of A.

$$Ax = 0$$

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ -2 \\ 1 \end{bmatrix} = 0$$

$$\underbrace{3a_1 - 2a_2 - 2a_3 + a_4 = 0}$$

This is a dependence relation among the columns of A.

(b) Explain why the existence of vector $\mathbf{x}$ in part (a) above tells you that the map T defined by $T(\mathbf{x}) = A\mathbf{x}$ is *not* one-to-one.

Several answers. Here are a few:

① Since $\vec{x} \neq 0$, $A\vec{x} = 0 \not\Rightarrow \vec{x} = 0$ so T is not 1-1

② Since $\vec{x} \neq 0$ but $A(\vec{x}) = A(0) = 0$, T is not 1-1

9. (8 points) Let $\mathcal{S} = \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ be a set of linearly independent vectors in $\mathbb{R}^m$.
Prove that the set of vectors $\mathcal{S}' = \{\mathbf{u}+\mathbf{v}, \mathbf{u}-\mathbf{v}, \mathbf{w}\}$ in $\mathbb{R}^m$ is also linearly independent.
The first line of your proof is: Suppose $x_1(\mathbf{u}+\mathbf{v}) + x_2(\mathbf{u}-\mathbf{v}) + x_3\mathbf{w} = \mathbf{0}_{\mathbb{R}^m}$

Proof. Suppose $x_1(u+v) + x_2(u-v) + x_3 w = 0_{\mathbb{R}^m}$

(we must show that $x_1 = x_2 = x_3 = 0$)

So: $x_1 u + x_1 v + x_2 u + x_2 v + x_3 w = 0_{\mathbb{R}^m}$

$*$ $(x_1 + x_2)u + (x_1 - x_2)v + x_3 w = 0_{\mathbb{R}^m}$

Since $S = \{u, v, w\}$ is linearly independent, we know that the only solution to $*$ is

$$\left.\begin{aligned} x_1 + x_2 &= 0 \\ x_1 - x_2 &= 0 \\ x_3 &= 0 \end{aligned}\right\} \qquad \begin{array}{r} x_1 + x_2 = 0 \\ + \quad x_1 - x_2 = 0 \\ \hline 2x_1 = 0 \\ x_1 = 0 \end{array}$$

and since $x_1 + x_2 = 0$ and $x_1 = 0$, $x_2 = 0$

$\therefore$ $x_1 = x_2 = x_3 = 0$ $\boxtimes$