KEY

Math 70
Linear Algebra

TUFTS UNIVERSITY
Department of Mathematics
Exam I

October 12, 2015
Sections 1 and 2

Instructions: No notes or books are allowed. All calculators, cell phones, or other electronic devices **must** be turned off and put away during the exam. Unless otherwise stated, you **must show all work** to receive full credit. *You are required to sign your exam. With your signature you are pledging that you have neither given nor received assistance on the exam. Students found violating this pledge will receive an F in the course.*

Problem	Point Value	Points
1	10	
2	20	
3	10	
4	8	
5	7	
6	7	
7	7	
8	7	
9	8	
10	8	
11	8	
	100	

1. (10 pts) **True/false questions.** Decide whether each of the statements below is true or false. Indicate your answer by shading the appropriate box. No partial credit.

   (a) Let T be a linear transformation. If $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent, then $\{T(\mathbf{u}), T(\mathbf{v}), T(\mathbf{w})\}$ is linearly independent. T F

   T could be the map $T(x) = 0$ for all x.

   (b) A consistent system of linear equations can have exactly 10 solutions. T F

   It has 1 or an infinite #

   (c) Every linear transformation from $\mathbb{R}^4$ to $\mathbb{R}^2$ is onto. T F

   T could be the map $T(x) = 0$ for all x.

   (d) The range of the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is the span of the columns of A. T F

   (e) For any 3×3 matrices A and B, we have $AB = BA$. T F

3×4

2. (20 pts) Let $A = \begin{bmatrix} 1 & 3 & -1 & 3 \\ 0 & 2 & 4 & 4 \\ 2 & 6 & -2 & 6 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 8 \\ 2 \end{bmatrix}$

(a) (2 pts) Write down the vector equation equivalent to $A\mathbf{x} = \mathbf{b}$.

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 2 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 4 \\ -2 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \\ 2 \end{bmatrix}$$

(b) (2 pts) Write down the linear system equivalent to $A\mathbf{x} = \mathbf{b}$.

$$x_1 + 3x_2 - x_3 + 3x_4 = 1$$
$$2x_2 + 4x_3 + 4x_4 = 8$$
$$2x_1 + 6x_2 - 2x_3 + 6x_4 = 2$$

(c) (6 pts) Find the parametric vector form of the solution set of $A\mathbf{x} = \mathbf{b}$.

$$\left(\begin{array}{cccc|c} 1 & 3 & -1 & 3 & 1 \\ 0 & 2 & 4 & 4 & 8 \\ 2 & 6 & -2 & 6 & 2 \end{array}\right)_{-2R_1} \sim \left(\begin{array}{cccc|c} 1 & 3 & -1 & 3 & 1 \\ 0 & 2 & 4 & 4 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \tfrac{\cdot}{2} \sim \left(\begin{array}{cccc|c} 1 & 3 & -1 & 3 & 1 \\ 0 & 1 & 2 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)^{-3R2}$$

$$\begin{array}{c} \begin{matrix} b & b & f & f & \end{matrix} \\ \left(\begin{array}{cccc|c} 1 & 0 & -7 & -3 & -11 \\ 0 & 1 & 2 & 2 & 4 \\ 0 & 0 & 0 & 0 & 6 \end{array}\right) \end{array}$$

$$x_1 - 7x_3 - 3x_4 = -11$$
$$x_2 + 2x_3 + 2x_4 = 4$$
$$x_3 = x_3$$
$$x_4 = x_4$$

$$x_1 = -11 + 7x_3 + 3x_4$$
$$x_2 = 4 - 2x_3 - 2x_4$$
$$x_3 = x_3$$
$$x_4 = x_4$$

$$\vec{x} = \begin{pmatrix} -11 \\ 4 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} 7 \\ -2 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 3 \\ -2 \\ 0 \\ 1 \end{pmatrix} \quad \text{where} \quad s, t \in \mathbb{R}$$

(d) (1 pt) Is the solution set of $A\mathbf{x} = \mathbf{b}$ a line in $\mathbb{R}^4$?

No - a plane.

(e) (2 pts) Let T_A be the linear transformation defined by $T_A(\mathbf{x}) = A\mathbf{x}$. What are the domain and codomain of T_A?

domain: $\mathbb{R}^4$ codomain $\mathbb{R}^3$

(f) (3 pts) Is T_A one-to-one? Explain why.

No T_A is 1-1 if and only if $Ax = 0$ has no free variables but here we have 2 free variables - x_3 and x_4; So $Ax = 0$ has non-trivial soln $\Rightarrow$ T_A not 1-1

(g) (4 pts) Is T_A onto? Explain why.

No - T_A is not onto because A does not have a pivot position in the 3rd row - so $Ax = b$ will not be consistent for all $b \in \mathbb{R}^3$.

3. (10 pts)

(a) Complete the following *definition*. A transformation (mapping) $T : \mathbb{R}^n \to \mathbb{R}^m$ is **linear** if

(i) $T(u+v) = T(u) + T(v)$ for all $u, v \in \mathbb{R}^n$

(ii) $T(cu) = cT(u)$ for all $u \in \mathbb{R}^n$ and scalars c.

(b) Use your definition to determine whether or not the following transformation is linear.

$$T : \mathbb{R}^3 \to \mathbb{R}^2, \quad T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_2 + x_3 \end{bmatrix}.$$

Let $u = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $v = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$ and c be a scalar

① $T(u+v) = T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}\right) = T\left(\begin{pmatrix} x+x' \\ y+y' \\ z+z' \end{pmatrix}\right) = \begin{pmatrix} (x+x') + (y+y') \\ (y+y') + (z+z') \end{pmatrix}$

$= \begin{pmatrix} (x+y) + (x'+y') \\ (y+z) + (y'+z') \end{pmatrix} = \begin{pmatrix} x+y \\ y+z \end{pmatrix} + \begin{pmatrix} x'+y' \\ y'+z' \end{pmatrix} = T(u) + T(v)$ $\boxed{X}$

② $T(cu) = T\left(c\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = T\left(\begin{pmatrix} cx \\ cy \\ cz \end{pmatrix}\right) = \begin{pmatrix} cx + cy \\ cy + cz \end{pmatrix} = c\begin{pmatrix} x+y \\ y+z \end{pmatrix} =$

$= cT(u).$ $\boxed{X}$

$\therefore$ T is linear.

4. (8 pts) It is a fact that $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 1 & -3 \\ 0 & 1 & -1 \end{bmatrix}$ are inverses of each other.

Use this fact to solve the system of equations:

$$\begin{aligned} - y + 2z &= 2 \\ x + y - 3z &= 5 \\ y - z &= 2 \end{aligned}$$

$$\begin{pmatrix} 0 & -1 & 2 \\ 1 & 1 & -3 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix}$$

$$B \quad \vec{x} \quad = \quad \vec{b}$$

$$\text{So. } \vec{x} = B^{-1}\vec{b} = A\vec{b}$$

$$= \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 11 \\ 6 \\ 4 \end{pmatrix}$$

5. (7 pts) Show that the following transformation $T : \cancel{\mathbb{R}^4}\ \mathbb{R}^3 \to \mathbb{R}^2$ is **not** linear by giving an explicit example (using specific vectors) that violates the definition of linear transformation.

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x^2 \\ y+z \end{bmatrix}.$$

<u>one example:</u>

$$u = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad v = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

Then $T(u+v) = T\left(\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 9 \\ 0 \end{pmatrix}$

but $T(u) + T(v) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$

Since $T(u+v) = \begin{pmatrix} 9 \\ 0 \end{pmatrix} \neq T(u) + T(v) = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$,

T is <u>not</u> linear.

6. (7 points) Let $A = \begin{bmatrix} -2 & 2 & 2 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix}$. Use the fact that $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ is a solution of $A\mathbf{x} = \mathbf{0}$ to construct a 3×3 matrix B with NO zero entry such that the AB is the zero matrix . Confirm your result. (Hint use $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ as the first column of B.)

Let $B = \begin{pmatrix} 2 & 2 & 2 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ Then $AB = \left(A\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}\ A\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}\ A\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

7. (7 pts) Determine whether each of the following sets in $\mathbb{R}^3$ is linearly independent. No explanation needed. Indicate your answer by shading the appropriate box. No partial credit.

(a) $S_1 = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}, \begin{bmatrix} 8 \\ 9 \\ 10 \end{bmatrix} \right\}$ $\boxed{\text{Yes}}$ $\boxed{\text{No}}$

Can't have > 3 L.I vectors in $\mathbb{R}^3$

If A =
(matrix whose columns contain the vectors,
$Ax=0$ will always have free variables,
hence $\exists$ non-0 solution to $Ax=0$)

(b) $S_2 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} \right\}$ $\boxed{\text{Yes}}$ $\boxed{\text{No}}$

(c) $S_3 = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} \right\}$ $\boxed{\text{Yes}}$ $\boxed{\text{No}}$

Contains $\vec{0}$

8. (7 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that rotates each point in $\mathbb{R}^2$ about the origin through $\frac{3\pi}{2}$ radians counterclockwise.

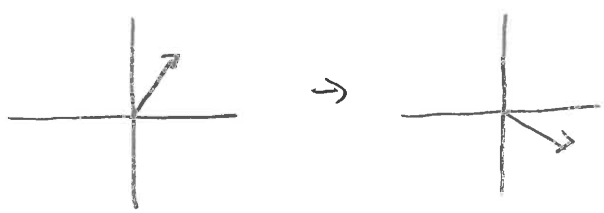
(a) Find the standard matrix A of T.

$$A = \begin{pmatrix} T(e_1) & T(e_2) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(b) Use the standard matrix to find $T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right)$.

$$T\left(\begin{pmatrix} 2 \\ 3 \end{pmatrix}\right) = A\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

ck:



9. (8 points) Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a map such that

$$T\left(\begin{bmatrix}2\\1\end{bmatrix}\right) = \begin{bmatrix}2\\1\end{bmatrix}, \quad T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = \begin{bmatrix}1\\2\end{bmatrix} \text{ and } T\left(\begin{bmatrix}3\\3\end{bmatrix}\right) = \begin{bmatrix}3\\4\end{bmatrix}.$$

Can T be a linear transformation? If your answer is yes, find the standard matrix of T. Otherwise, explain why T can not be a linear transformation.

Let $u = \begin{pmatrix}2\\1\end{pmatrix}$, $v = \begin{pmatrix}1\\2\end{pmatrix}$ and $w = \begin{pmatrix}3\\3\end{pmatrix}$

Notice that $u+v = w$ So $T(u+v) = T(w)$ must equal $T(u) + T(v)$ if T is to be linear.

$T(u+v) = T(w) = \begin{pmatrix}3\\4\end{pmatrix}$

$T(u)+T(v) = \begin{pmatrix}2\\1\end{pmatrix} + \begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}3\\3\end{pmatrix}$

Since $T(u+v) \neq T(u)+T(v)$, T is not linear.

$\left(\begin{pmatrix}3\\4\end{pmatrix} \neq \begin{pmatrix}3\\3\end{pmatrix} \right)$

10. (8 points) Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be vectors in $\mathbb{R}^3$. Suppose $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ spans $\mathbb{R}^3$. Is $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly independent? Explain your answer.

Let $A = (v_1\ v_2\ v_3)$. Then A is 3×3.

Since S spans $\mathbb{R}^3$ we know that A has a pivot position in every row. But since A is 3×3, it also has a pivot position in every column. Thus $Ax=0$ has no free variables $\Rightarrow$ the only solution to $Ax=0$ is $x=0$. Thus the set S is LI

11. (8 points) Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be vectors in $\mathbb{R}^n$. Suppose $\mathbf{v}_3$ is in the span of $\mathbf{v}_1, \mathbf{v}_2$, i.e. $\mathbf{v}_3 \in \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. Prove that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent. Justify each step of your proof.

Since $v_3 \in \text{Span}\{v_1, v_2\}$, there exist scalars c and d

such that $v_3 = c_1 v_1 + dv_2$

or, $c_1 v_1 + dv_2 - v_3 = \mathbf{0}$.

This is a dependence relation because c, d, and -1 are not all 0.

$\therefore$ $\{v_1, v_2, v_3\}$ is linearly dependent. $\boxed{X}$

Scratch work

Math 70 Exam I October 12, 2015 Sections 1 and 2

Name ________________________________

Please circle your section

Section 1 Haio Liang

Section 2 Mary Glaser

I pledge that I have neither given nor received assistance on this exam.

Signature ________________________________