Math 46 Exam I Fall 2006 SOLUTIONS

1. (10) $T: \mathbb{R}^2 \to \mathbb{R}^3$ $T(x_1, x_2) = (2x_1 + x_2, -3x_2, 0)$

i) Let $\underline{u} = (x_1, x_2)$, $\underline{v} = (y_1, y_2) \in \mathbb{R}^2$

$$T(\underline{u}+\underline{v}) = T((x_1,x_2)+(y_1,y_2)) = T(x_1+y_1, x_2+y_2) = (2(x_1+y_1) + (x_2+y_2), -3(x_2+y_2), 0)$$

$$= (2x_1 + 2y_1 + x_2 + y_2, -3x_2 - 3y_2, 0) = (2x_1 + x_2, -3x_2, 0) + (2y_1 + y_2, -3y_2, 0)$$

$$= T(\underline{u}) + T(\underline{v}) \boxtimes$$

ii) Let $\underline{u} = (x_1, x_2) \in \mathbb{R}^2$, c is a scalar

$$T(c\underline{u}) = T(c(x_1,x_2)) = T(cx_1, cx_2) = (2cx_1 + cx_2, -3cx_2, 0) = c(2x_1 + x_2, -3x_2, 0) = cT(\underline{u}) \boxtimes$$

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2. (15) $\underline{u} = (1,1)$, $\underline{v} = (3,2)$, $\underline{w} = (2,0)$

a) $\underline{w} \in \text{span}\{\underline{u}, \underline{v}\}$: $\left[\begin{array}{cc|c} 1 & 3 & 2 \\ 1 & 2 & 0 \end{array}\right] \xrightarrow{-R_1} \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & -1 & -2 \end{array}\right] \xrightarrow{+3R2} \left[\begin{array}{cc|c} 1 & 0 & -4 \\ 0 & -1 & -2 \end{array}\right] \xrightarrow{\cdot -1} \left[\begin{array}{cc|c} 1 & 0 & -4 \\ 0 & 1 & 2 \end{array}\right]$

so $\underline{w} = -4\underline{u} + 2\underline{v}$.

b) $T(\underline{w}) = T(-4\underline{u} + 2\underline{v}) = -4T(\underline{u}) + 2T(\underline{v}) = -4(-3,4) + 2(-1,6) = (10,-4)$

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3. (15) a) Force $A\underline{x} = \underline{0}$ to have free variables

$$\begin{array}{ccc} x_1 & x_2 & x_3 \end{array}$$
$$\begin{bmatrix} 1 & -5 & 1 \\ -1 & 7 & 1 \\ -3 & 8 & h \end{bmatrix} \begin{array}{l} \\ +R_1 \\ +3R_1 \end{array} \begin{bmatrix} 1 & -5 & 1 \\ 0 & 2 & 2 \\ 0 & -7 & h+3 \end{bmatrix} \tfrac{1}{2} \begin{bmatrix} 1 & -5 & 1 \\ 0 & 1 & 1 \\ 0 & -7 & h+3 \end{bmatrix} + 7R2 \begin{bmatrix} 1 & -5 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & h+10 \end{bmatrix}$$

To force x_3 to be free set $h = -10$.

b) $\left[\begin{array}{ccc|c} 1 & -5 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{+5R2} \left[\begin{array}{ccc|c} 1 & 0 & 6 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$
$x_1 = -6x_3$
$x_2 = -x_3$
$x_3 = x_3$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -6 \\ -1 \\ 1 \end{bmatrix}$$

one example: $\boxed{-6\underline{v}_1 - \underline{v}_2 + \underline{v}_3 = \underline{0}}$ (check)

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4. (10) If $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ in $\mathbb{R}^5$ are linearly dependent $\to \exists$ constants c_1, c_2, c_3 not all 0 with

$$c_1\underline{v}_1 + c_2\underline{v}_2 + c_3\underline{v}_3 = \underline{0}$$

But now $c_1\underline{v}_1 + c_2\underline{v}_2 + c_3\underline{v}_3 + 0\cdot\underline{v}_4 = \underline{0}$ and since we know c_1, c_2, c_3 are not all 0, this is a dependency relation among $\{\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4\} \to \{\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4\}$ is linearly dependent.

5. (15) a) $T: \mathbb{R}^5 \to \mathbb{R}^m$ is onto $\to$ $m = 1, 2, 3, 4,$ or 5.

b) $T: \mathbb{R}^n \to \mathbb{R}^7$ 1-1 $\to$ $n = 1, 2, 3, 4, 5, 6,$ or 7.

c) $T: \mathbb{R}^7 \to \mathbb{R}^8$ is 1-1. $T(\underline{x}) = A\underline{x}$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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6. (10) $\begin{pmatrix} 1 & 5 & 2 & -6 & 9 & 0 \\ 0 & 0 & 1 & -7 & 4 & -8 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} -2R2 \quad \begin{pmatrix} 1 & 5 & 0 & 8 & 1 & 16 \\ 0 & 0 & 1 & -7 & 4 & -8 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} -16R3 \\ +8R3 \\ \\ \\ \end{matrix} \quad \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ \end{array} \begin{pmatrix} 1 & 5 & 0 & 8 & 1 & 0 \\ 0 & 0 & 1 & -7 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

$\uparrow \quad \uparrow \uparrow$ free

$x_1 + 5x_2 + 8x_4 + x_5 = 0$
$x_2 = x_2$
$x_3 - 7x_4 + 4x_5 =$
$x_4 = x_4$
$x_5 = x_5$
$x_6 = 0$

$x_1 = -5x_2 - 8x_4 - x_5$
$x_2 = x_2$
$x_3 = 7x_4 - 4x_5$
$x_4 = x_4$
$x_5 = x_5$
$x_6 = 0$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = x_2 \begin{bmatrix} -5 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -8 \\ 0 \\ 7 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ 0 \\ -4 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

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7. (10) A is $n \times n$ and $A\underline{x} = \underline{0}$ has non-trivial solutions $\to$ $A\underline{x} = \underline{0}$ has free variables so there is not a pivot position in every column. Hence, since A is square there is not a pivot position in every row. Thus $A\underline{x} = \underline{b}$ is not consistent for every choice of $\underline{b}$ $\to$ $T(\underline{x}) = A\underline{x}$ is not onto.

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8. a) (4) Suppose $\{\underline{v}_1, \underline{v}_2, \underline{v}_3, \ldots \underline{v}_p\}$ is linearly independent in $\mathbb{R}^n$ and $T: \mathbb{R}^n \to \mathbb{R}^m$ is 1-1.
Show $\{T(\underline{v}_1), T(\underline{v}_2), \ldots T(\underline{v}_p)\}$ is linearly independent.

pf Let $x_1 T(\underline{v}_1) + x_2 T(\underline{v}_2) + \cdots + x_p T(\underline{v}_p) = 0$. Since T is linear we can rewrite this as:
$T(x_1 \underline{v}_1 + x_2 \underline{v}_2 + \cdots x_p \underline{v}_p) = \underline{0}$. Since T is 1-1, this equation has only the trivial solution, i.e. $x_1 \underline{v}_1 + x_2 \underline{v}_2 + \cdots + x_p \underline{v}_p = \underline{0}$. But since $\{\underline{v}_1, \underline{v}_2, \underline{v}_3, \ldots, \underline{v}_p\}$ is linearly independent, $x_1 = x_2 = \cdots = x_p = 0$. $\boxtimes$

(b) i) $p \le n$ T
ii) $p > n$ F
iii) $p \le m$ T
iv) $p > m$ F
v) $n \le m$ T
vi) $n > m$ F