Math 70	**TUFTS UNIVERSITY**	**Feb 26, 2018**
Linear Algebra	**Department of Mathematics**	**Sections 2,3,4**
		Exam 1

Instructions: No notes or books are allowed. All calculators, cell phones, or other electronic devices **must** be turned off and put away during the exam. Unless otherwise stated, you **must show all work** to receive full credit. *You are required to sign your exam. With your signature you are pledging that you have neither given nor received assistance on the exam. Students found violating this pledge will receive an F in the course.*

Problem	Point Value	Points
1	20	
2	10	
3	10	
4	8	
5	10	
6	14	
7	12	
8	8	
9	8	
	100	

1. (20 points) Consider the following system of equations:

$$x_1 - 2x_2 + 2x_3 + 10x_4 = 3$$
$$x_1 - 2x_2 + 5x_3 + 25x_4 = 9$$

(a) Write the system as a a vector equation.

(b) Write the system as a matrix equation $A\mathbf{x} = \mathbf{b}$.

(c) Solve the system and write your solution in parametric vector form.

2. (10 pts) **True/false questions.** For each of the statements below, decide whether it is true or false. Indicate your answer by shading the corresponding box. There will be no partial credit.

(a) The map $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(\mathbf{v}) = \mathbf{0}_{\mathbb{R}^m}$ for all $\mathbf{v}$ in $\mathbb{R}^n$ is linear. T F

(b) Let A be an $m \times n$ matrix.
The range of the map $\mathbf{x} \mapsto A\mathbf{x}$ is the span of the columns of A. T F

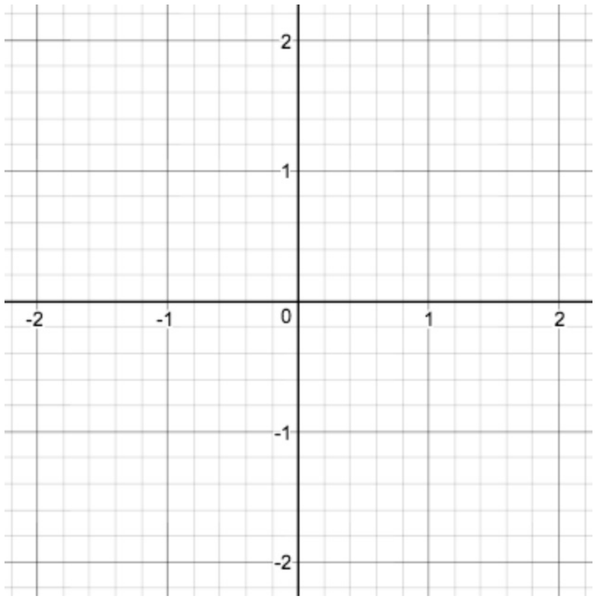
(c) Every set of four vectors in $\mathbb{R}^3$ is linearly dependent. T F

(d) Every linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$ is onto. T F

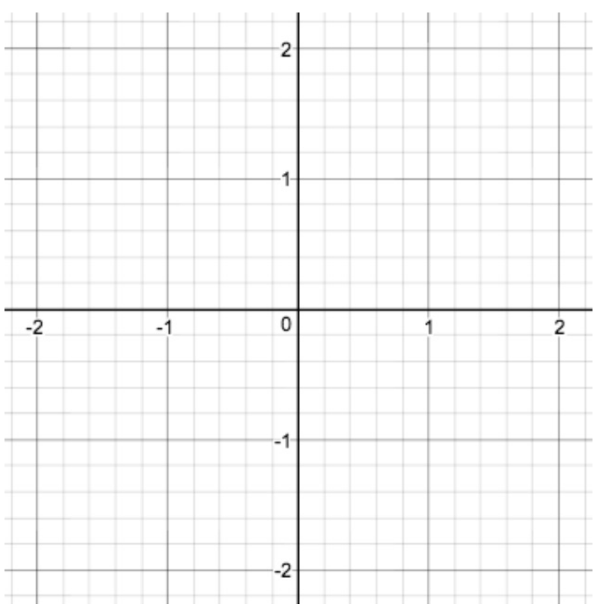
(e) If a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent, then one of the vectors is a scalar multiple of one of the others. T F

3. (10 points) Plot the images of $\mathbf{e}_1$ and $\mathbf{e}_2$ under the following linear transformations and then write down the standard matrix for each transformation.

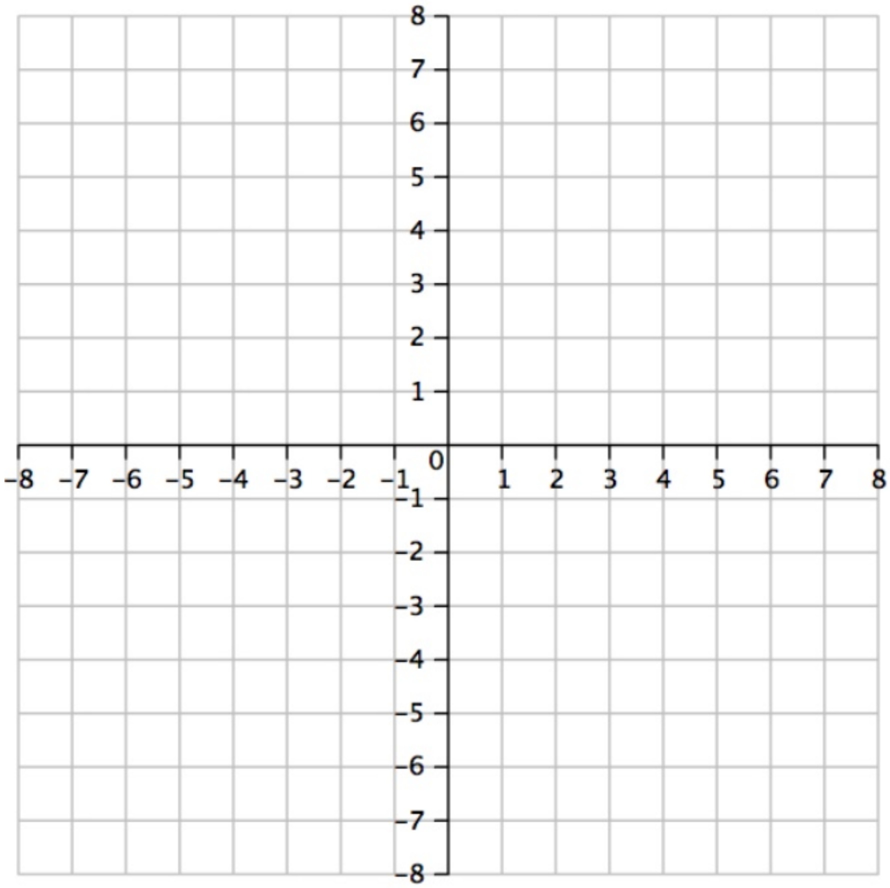
(a) $T : \mathbb{R}^2 \to \mathbb{R}^2$ rotates points (about the origin) $\pi/4$ radians.



(b) $T : \mathbb{R}^2 \to \mathbb{R}^2$ first reflects points through the horizontal x_1-axis and then reflects points across the line $x_1 = x_2$.



4. (8 points) Let A be a 2×2 matrix. Suppose the solution set to the homogeneous equation $A\mathbf{x} = \mathbf{0}$ is $\{t\mathbf{v} \mid \text{t is any scalar}\}$ where $\mathbf{v} = (4, 2)$. Suppose that $\mathbf{p} = (-3, 2)$ is a particular solution to the nonhomogeneous equation $A\mathbf{x} = \mathbf{b}$ for the same matrix A and for some nonzero vector $\mathbf{b}$. Sketch and clearly label the solution sets to $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{x} = \mathbf{b}$ on the grid below.



5. (10 points)

   (a) Write down a matrix A in row echelon form that represents a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ that is onto but not one-to-one. Briefly explain why your matrix is correct.

   (b) Write down a matrix B in row echelon form that represents a linear transformation $S : \mathbb{R}^3 \to \mathbb{R}^5$ that is one-to-one but not onto. Briefly explain why your matrix is correct.

6. (14 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $T(x, y) = (3x, x - 2y, 0)$.

   (a) Write down the *definition* of what it means to say that $T : \mathbb{R}^2 \to \mathbb{R}^3$ is **linear**.

   (b) Use your definition to show that T is linear. Show all steps.

7. (12 points)
   (a) Write the augmented matrix of the following linear system and row reduce it to echelon form.

$$x_1 + 5x_2 = k$$
$$4x_1 + hx_2 = 12$$

   (b) Find all value(s) of h and k so the linear system above has
      (i) no solutions

      (ii) only one solution

      (iii) infinitely many solutions

8. (8 points) Let A be an $m \times 4$ matrix with columns $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$, i.e. $A = [\, \mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3\ \mathbf{a}_4]$.

   (a) Suppose $\mathbf{x} = \begin{bmatrix} 3 \\ -2 \\ -2 \\ 1 \end{bmatrix}$ is a solution to the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

   Write down a dependence relation among the columns of A.

   (b) Explain why the existence of vector $\mathbf{x}$ in part (a) above tells you that the map T defined by $T(\mathbf{x}) = A\mathbf{x}$ is *not* one-to-one.

9. (8 points) Let $\mathcal{S} = \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ be a set of linearly independent vectors in $\mathbb{R}^m$.
Prove that the set of vectors $\mathcal{S}' = \{\mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v}, \mathbf{w}\}$ in $\mathbb{R}^m$ is also linearly independent.
The first line of your proof is: Suppose $x_1(\mathbf{u} + \mathbf{v}) + x_2(\mathbf{u} - \mathbf{v}) + x_3\mathbf{w} = \mathbf{0}_{\mathbb{R}^m}$

Math 70 Exam I February 26, 2018

Name: ____________________________________

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Section 3 Mary Glaser T,Th,F 12-12:50

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I pledge that I have neither given nor received assistance on this exam.

Signature ____________________________________