Math 70 | **TUFTS UNIVERSITY** | **October 7, 2013**

Linear Algebra | **Department of Mathematics** | **Section 1**

Exam I

Instructions: No notes or books are allowed. All calculators, cell phones, or other electronic devices **must** be turned off and put away during the exam. Unless otherwise stated, you **must show all work** to receive full credit. *You are required to sign your exam. With your signature you are pledging that you have neither given nor received assistance on the exam. Students found violating this pledge will receive an F in the course.*

Problem	Point Value	Points
1	10	
2	20	
3	12	
4	6	
5	10	
6	8	
7	9	
8	8	
9	10	
10	7	
	100	

1. (10 pts) **True/false questions.** Decide whether each of the statements below is true or false. Indicate your answer by shading the appropriate box. No partial credit.

   (a) If $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent and $\mathbf{w}$ is in Span$\{\mathbf{u}, \mathbf{v}\}$, then $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent. [T] [F]

   (b) Every matrix transformation is linear. [T] [F]

   (c) If A is a 3×2 matrix, the transformation $\mathbf{x} \mapsto A\mathbf{x}$ cannot be one-to-one. [T] [F]

   (d) A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if every vector $\mathbf{x}$ in $\mathbb{R}^n$ maps to a unique vector $\mathbf{y}$ in $\mathbb{R}^m$. [T] [F]

   (e) If $A\mathbf{x} = \mathbf{b}$ is consistent, then the solution set of $A\mathbf{x} = \mathbf{b}$ is obtained by translating the solution set of $A\mathbf{x} = \vec{0}$. [T] [F]

2. (20 pts) Consider the system of equations:

$$5x_1 - 5x_2 + 10x_3 + 15x_5 = 20$$

$$2x_1 - 2x_2 + 4x_3 + x_4 + 2x_5 = 3$$

(a) Write down the corresponding vector equation.

(b) Write down the corresponding matrix equation and label it $A\mathbf{x} = \mathbf{b}$.

(c) The linear transformation T with $T(x) = Ax$ represents a map from $\mathbb{R}^n$ to $\mathbb{R}^m$ where

n = ________________ and m = ________________.

(d)

$$5x_1 - 5x_2 + 10x_3 + 15x_5 = 20$$
$$2x_1 - 2x_2 + 4x_3 + x_4 + 2x_5 = 3$$

Find all solutions to the system and write your answer in parametric vector form.

3. (12 pts) Fill in the blanks.

   (a) A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is called *onto* if for every $\mathbf{b}$ in ____________ there is

   ____________________ one $\mathbf{x}$ in ____________ such that ____________________.

   If T is onto and $T(\mathbf{x}) = A\mathbf{x}$ the columns of A ____

   and A has a pivot position in every ________________.

   (b) A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is called *one-to-one* if for every $\mathbf{b}$ in ____________

   there is ____________________ one $\mathbf{x}$ in ____________ such that ____________________.

   If T is one-to-one and $T(\mathbf{x}) = A\mathbf{x}$ the columns of A ____

   and A has a pivot position in every ________________.

4. (6 pts)

   (a) Suppose $T : \mathbb{R}^4 \to \mathbb{R}^m$ is an onto linear transformation. List all possible values for m.

   (b) Suppose $T : \mathbb{R}^n \to \mathbb{R}^6$ is a one-to-one linear transformation. List all possible values for n.

5. (10 points) Let $\mathbf{u} = \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \vdots \\ \mathbf{u}_n \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_n \end{bmatrix}$ be vectors in $\mathbb{R}^n$. Show that if c is a scalar then $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.

6. (8 pts) Show why **neither** of the following transformations is linear. Be as explicit as possible.

(a) $T : \mathbb{R}^3 \to \mathbb{R}^2$, $T(x, y, z) = (x - 2, z)$.

(b) $T : \mathbb{R}^2 \to \mathbb{R}^2$, $T(x, y) = (x, xy)$.

7. (9 pts) Determine whether each of the following sets in $\mathbb{R}^3$ is linearly independent. No explanation needed. Indicate your answer by shading the appropriate box. No partial credit.

(a) $S_1 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$. Yes No

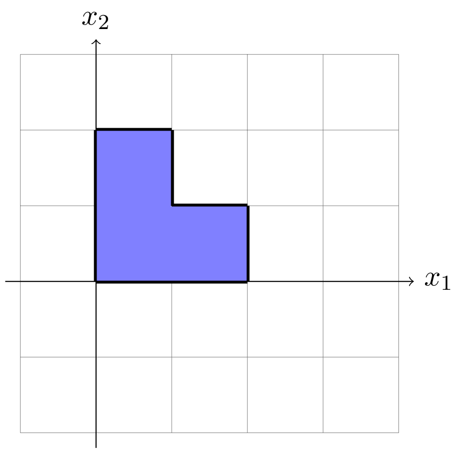
(b) $S_2 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -3 \end{bmatrix} \right\}$. Yes No

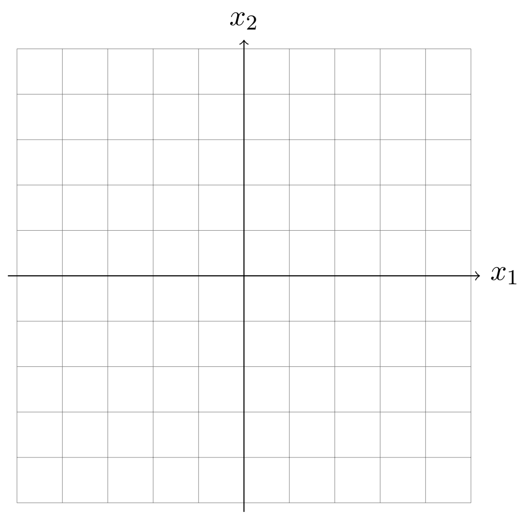
(c) $S_3 = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$. Yes No

8. (8 points)

(a) Here is the matrix of a linear transformation: $A = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}$.

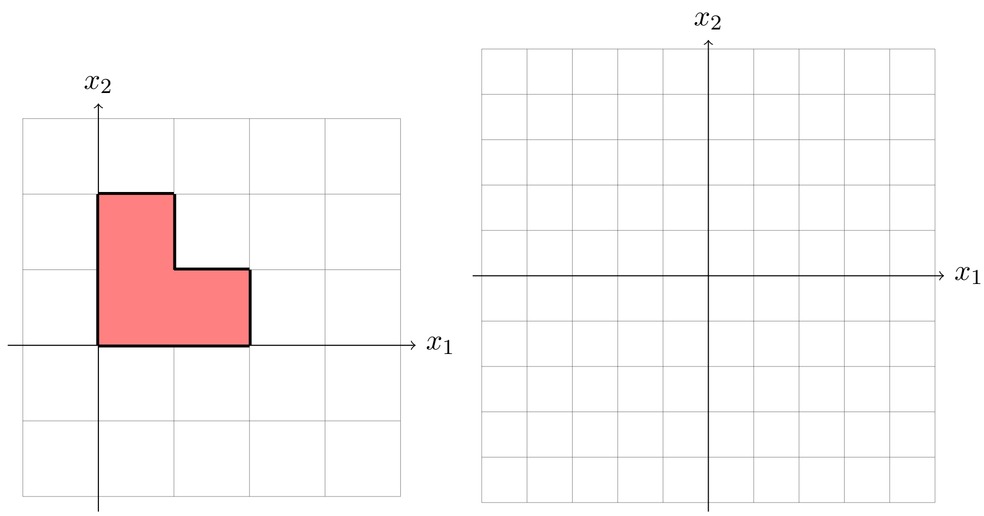
Draw the image of the figure below under the transformation $T(\mathbf{x}) = A\mathbf{x}$..





(b) Here is the matrix of a linear transformation: $A = \begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix}$.

Draw the image of the figure below under the transformation.



9. (10 points) Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a linear transformation.

(a) Define the standard matrix A of T.

(b) Suppose that

$$T\left(\begin{bmatrix} 3 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -12 \\ 6 \end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix} 3 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

Use your definition from part (a) along with the definition of a linear transformation to find the standard matrix A of T.

10. (7 points) Suppose $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ span $\mathbb{R}^3$ and let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation. Suppose $T(\mathbf{v}_i) = \vec{0}$, for $i = 1, 2, 3, 4$. Show that T is the zero transformation. That is, show that if $\mathbf{x}$ is any vector in $\mathbb{R}^3$, then $T(\mathbf{x}) = \vec{0}$. (Your answer should not involve matrices.)

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Name ____

I pledge that I have neither given nor received assistance on this exam.

Signature ____