Tufts University

Math 46 (sections 01 and 02) Department of Mathematics

October 11, 2006

Exam 1

No calculators, notes, books, pagers, mobile phones or other electronic devices are allowed on the exam. You must **show all your work** to receive credit. Justify all your answers. Cross out anything you do not want graded. *You are required to sign your exam book. With your signature you are pledging that you have neither given nor received assistance on the exam. Students found violating this pledge will receive an F in the course.*

1. (10 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $T(x_1, x_2) = (2x_1 + x_2, -3x_2, 0)$. Prove that T is a linear transformation.

2. (15 points)

   Let $\mathbf{u} = (1,1), \mathbf{v} = (3,2)$ and $\mathbf{w} = (2,0)$

   (a) Show that $\mathbf{w}$ is in Span$\{\mathbf{u}, \mathbf{v}\}$

   (b) Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation with $T(\mathbf{u}) = (-3,4)$ and $T(\mathbf{v}) = (-1,6)$. Find $T(\mathbf{w})$.

3. (15 points) Let $A = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] = \begin{bmatrix} 1 & -5 & 1 \\ -1 & 7 & 1 \\ -3 & 8 & h \end{bmatrix}$

   (a) Find all values of h for which $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent.

   (b) Find a specific dependency relation among $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

4. (10 points)

   Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ be four vectors in $\mathbb{R}^5$. Assume $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent. Show that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly dependent.

5. (15 points)

   (a) Let $T : \mathbb{R}^5 \to \mathbb{R}^m$ be an <u>onto</u> linear transformation. List all possible values of m.

   (b) Let $T : \mathbb{R}^n \to \mathbb{R}^7$ be a <u>one-to-one</u> linear transformation. List all possible values of n.

   (c) Let $T : \mathbb{R}^7 \to \mathbb{R}^8$ be <u>one-to-one</u> and assume $T(\mathbf{x}) = A\mathbf{x}$. List all possible reduced echelon forms of A.

EXAM CONTINUES ON OTHER SIDE

6. (10 points) Describe all solutions of $A\mathbf{x} = \mathbf{0}$ where A is the matrix

$$\begin{pmatrix} 1 & 5 & 2 & -6 & 9 & 0 \\ 0 & 0 & 1 & -7 & 4 & -8 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

7. (10 points) Suppose A is an n x n matrix with the property that the equation $A\mathbf{x} = \mathbf{0}$ has non-trivial solutions. Explain, without quoting the invertible matrix theorem, why $T(\mathbf{x}) = A\mathbf{x}$ is not onto.

8. (a) (9 points) If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ in $\mathbb{R}^n$ are linearly independent vectors and if $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one, prove that $T(\mathbf{v}_1), \ldots, T(\mathbf{v}_p)$ are linearly independent vectors in $\mathbb{R}^m$. (Hint: Your proof should start "Let $x_1 T(\mathbf{v}_1) + \ldots + x_p T(\mathbf{v}_p) = \mathbf{0}$" and conclude "then $x_1 \ldots = x_p = 0$ which proves the result").

   (b) (6 points) Which of the following are true or false

      i. $p \leq n$
      ii. $p > n$
      iii. $p \leq m$
      iv. $p > m$
      v. $n \leq m$
      vi. $n > m$