Section 6.4 – The Gram-Schmidt Process

Main Idea in this section:

- Building orthogonal bases out of non-orthogonal bases

Exercises

1. Find the component of $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ that's orthogonal to $\begin{bmatrix} -7 \\ 5 \end{bmatrix}$, then find the component of $\begin{bmatrix} -7 \\ 5 \end{bmatrix}$ that's orthogonal to $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

2. For the subspace $span \left\{ \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -7 \\ 5 \end{bmatrix} \right\}$, find:

(a) a non-orthogonal basis

(b) two orthogonal bases

(c) an orthonormal basis

3. Let $\mathbf{x}_1 = \begin{bmatrix} 4 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 2 \end{bmatrix}$, and let $W = span\{\mathbf{x}_1, \mathbf{x}_2\}$.

   Describe a way to find an orthogonal basis for W.

4. Find an orthogonal basis for $span\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, where $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \in \mathbb{R}^n$ are linearly independent.

Theorem 11 The Gram-Schmidt Process
Given a basis $\{\mathbf{x}_1, \dots, \mathbf{x}_p\}$ for a subspace W of $\mathbb{R}^n$, define

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{x}_1 \\
\mathbf{v}_2 &= \mathbf{x}_2 - \left(\frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 \\
\mathbf{v}_3 &= \mathbf{x}_3 - \left(\frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2 \\
&\vdots \\
\mathbf{v}_p &= \mathbf{x}_p - \left(\frac{\mathbf{x}_p \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{x}_p \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2 - \cdots - \left(\frac{\mathbf{x}_p \cdot \mathbf{v}_{p-1}}{\mathbf{v}_{p-1} \cdot \mathbf{v}_{p-1}}\right)\mathbf{v}_{p-1}.
\end{aligned}$$

Then $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is an orthogonal basis for W. In addition, $span\{\mathbf{v}_1, \dots, \mathbf{v}_k\} = span\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$ for $1 \leq k \leq p$.

5. Find an orthogonal basis for

$$span \left\{ \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 3 \\ 6 \end{bmatrix} \right\}.$$

Theorem 12 QR Factorization If A is an $m \times n$ matrix with linearly independent columns, then A can be factored as $A = QR$, where Q is an $m \times n$ matrix whose columns form an orthonormal basis for $col(A)$ and R is an $n \times n$ upper triangular invertible matrix with positive entries on its diagonal.

Closer look at Q and R:

Example: finding the QR factorization of a matrix

- Start with matrix $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 0 & 3 \end{bmatrix}$. (LI columns)
- **Gram-Schmidt** process plus normalization yields $\left\{ \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$, an orthonormal basis for $col(A) = span \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} \right\}$.
- Then $Q = \begin{bmatrix} 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 \\ 0 & 1 \end{bmatrix}$.
- Q was built to be orthogonal, so $Q^T Q = I$.
- So if $A = QR$, then ...

- Computationally,

$$R = Q^T A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 2\sqrt{2} \\ 0 & 3 \end{bmatrix}.$$

Summary

- $\mathbf{y} - \left(\frac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u} =$ component of $\mathbf{y}$ orthogonal to $\mathbf{u}$.
- $\mathbf{y} - \left(\frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\right)\mathbf{u}_1 - \cdots - \left(\frac{\mathbf{y}\cdot\mathbf{u}_p}{\mathbf{u}_p\cdot\mathbf{u}_p}\right)\mathbf{u}_p =$ component of $\mathbf{y}$ orthogonal to $\mathbf{u}_1, \ldots, \mathbf{u}_p$.
- Gram-Schmidt Process: to build an orthogonal basis from a non-orthogonal set of vectors, subtract the orthogonal projections (onto every current vector in the set) from each new vector added.
- Any matrix A with LI columns can be factored into $A = QR$, with Q an orthogonal matrix and R upper triangular and invertible.