Section 6.2, 6.3 – Orthogonal Sets, Orthogonal Projections

Main Ideas in these sections:

- Orthogonal set = A set of mutually orthogonal vectors.
- OG $\Rightarrow$ LI.
- Orthogonal Projection of $\mathbf{y}$ onto $\mathbf{u}$ or onto an OG set $\{\mathbf{u}_1, \cdots, \mathbf{u}_p\}$; geometry, computation, interpretation.
- Expressing $\mathbf{y}$ in terms of an OG basis.
- Projection of $\mathbf{y}$ onto subspaces W and $W^{\perp}$.
- Orthonormal sets, Orthogonal matrices

- A set of vectors $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ in $\mathbb{R}^n$ is an **orthogonal set** if $\mathbf{u}_i \cdot \mathbf{u}_j = 0$ whenever $i \neq j$.

Examples

- The set $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \right\}$ is an OG set because $\mathbf{u}_1 \cdot \mathbf{u}_2 = 0, \quad \mathbf{u}_1 \cdot \mathbf{u}_3 = 0, \quad \mathbf{u}_2 \cdot \mathbf{u}_3 = 0.$

- The set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} \right\}$ is *not* an OG set because $\mathbf{v}_3$ is not OG to $\mathbf{v}_1$ or $\mathbf{v}_2$.

- **OG $\Rightarrow$ LI.**

Theorem 4 Suppose $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is an OG set of nonzero vectors in $\mathbb{R}^n$. Then S is a linearly independent set. Thus, if $W = span\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$, S is a basis for W.

Proof that OG $\Rightarrow$ LI

Examine the condition for linear independence. That is, find all combinations of c_i's that make a true statement out of:

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_p\mathbf{u}_p = \mathbf{0}.$$

Question: What's the difference between LI vectors and OG vectors?

$\star$ An **orthogonal basis** for a subspace W of $\mathbb{R}^n$ is a basis for W that is also an orthogonal set.

• OG Projection of $\mathbf{y}$ onto $\mathbf{u}$; onto $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$

Background: Let $\mathbf{y}, \mathbf{u}$ be nonzero, unequal vectors in $\mathbb{R}^n$. Express $\mathbf{y}$ as

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z},$$

where $\hat{\mathbf{y}}$ is a vector in the same direction as $\mathbf{u}$ and $\mathbf{z}$ is perpendicular to $\mathbf{u}$.

⋆The vector $\hat{\mathbf{y}}$ in the example above is called the **orthogonal projection of y onto u**, and is denoted $\text{proj}_{\mathbf{u}}\mathbf{y}$.

⋆The vector $\mathbf{z}$ in the example above is called the **component of y orthogonal to u**.

Exercises

1. For $\mathbf{y} = \begin{bmatrix} 8 \\ -4 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, find the orthogonal projection of $\mathbf{y}$ onto $\mathbf{u}$. Then write $\mathbf{y}$ as the sum of a vector in $span\{\mathbf{u}\}$ and a vector orthogonal to $\mathbf{u}$.

2. For $\mathbf{y} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$, find the point in $span\{\mathbf{u}\}$ that's closest to $\mathbf{y}$.

3. For the problem above, find the distance from $\mathbf{y}$ to $span\{\mathbf{u}\}$

4. Let $\mathbf{y} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$, $\mathbf{u}_1 = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$, and $\mathbf{u}_2 = \begin{bmatrix} 4 \\ -4 \end{bmatrix}$.

(a) Prove that $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an OG basis for $\mathbb{R}^2$.

(b) Sketch $\mathbf{y}, \mathbf{u}_1, \mathbf{u}_2$, $\text{proj}_{\mathbf{u}_1}\mathbf{y}$, and $\text{proj}_{\mathbf{u}_2}\mathbf{y}$.

(c) What is $\mathbf{y} - (\text{proj}_{\mathbf{u}_1}\mathbf{y} + \text{proj}_{\mathbf{u}_2}\mathbf{y})$? Explain.

Solution:

(a) $\mathbf{u}_1 \cdot \mathbf{u}_2 = 0$, so $\mathbf{u}_1, \mathbf{u}_2$ are OG, hence LI. Two LI vectors in $\mathbb{R}^2$ form a basis of $\mathbb{R}^2$. (Could also use Thm. 4)

(b)

(c) $\mathbf{y} - (\text{proj}_{\mathbf{u}_1}\mathbf{y} + \text{proj}_{\mathbf{u}_2}\mathbf{y}) = \mathbf{0}$ because projection onto *all* basis vectors captures *every* part of $\mathbf{y}$.

• Expressing $\mathbf{y}$ in terms of an OG basis

Theorem 5 Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ be an *orthogonal* basis for a subspace W of $\mathbb{R}^n$. For each $\mathbf{y} \in W$, the unique weights $(c_1,\ c_2,\ \ldots,\ c_p)$ in the linear combination

$$\mathbf{y} = c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p$$

are given by

$$c_j = \frac{\mathbf{y} \cdot \mathbf{u}_j}{\mathbf{u}_j \cdot \mathbf{u}_j} \qquad \text{for} \quad j = 1 \cdots p.$$

5. Express $\mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ as a linear combination of the vectors in the orthogonal basis

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \right\} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \text{ of } \mathbb{R}^3.$$

Solution:

$$\begin{aligned} \mathbf{y} &= \text{proj}_{\mathbf{v}_1}\mathbf{y} + \text{proj}_{\mathbf{v}_2}\mathbf{y} + \text{proj}_{\mathbf{v}_3}\mathbf{y} \\ &= \left(\frac{\mathbf{y} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 + \left(\frac{\mathbf{y} \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2 + \left(\frac{\mathbf{y} \cdot \mathbf{v}_3}{\mathbf{v}_3 \cdot \mathbf{v}_3}\right)\mathbf{v}_3 \\ &= (-1/2)\mathbf{v}_1 + (1/3)\mathbf{v}_2 + (2)\mathbf{v}_3 \end{aligned}$$

- **Projection of $\mathbf{y}$ onto subspaces W and $W^\perp$**

6. Given $\mathbf{y} = \begin{bmatrix} 0 \\ 3 \\ 10 \end{bmatrix}$ and the OG basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \left\{ \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -6 \end{bmatrix} \right\}$ of $\mathbb{R}^3$. Define the subspace W by $W = span\{\mathbf{v}_1, \mathbf{v}_2\}$. Find the component of $\mathbf{y}$ that lies within W. [Hint: use theorem 5]

7. For the problem above, find the distance from $\mathbf{y}$ to W.

Thm 8: The Orthogonal Decomposition Theorem
Let W be a subspace of $\mathbb{R}^n$. Then each $\mathbf{y} \in \mathbb{R}^n$ can be written uniquely in the form

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$$

where $\hat{\mathbf{y}} \in W$ and $\mathbf{z} \in W^{\perp}$. In fact, if $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is any orthogonal basis of W, then

$$\hat{\mathbf{y}} = \left(\frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\right)\mathbf{u}_1 + \left(\frac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\right)\mathbf{u}_2 + \cdots + \left(\frac{\mathbf{y} \cdot \mathbf{u}_p}{\mathbf{u}_p \cdot \mathbf{u}_p}\right)\mathbf{u}_p$$

and $\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}$.

Remarks:

- The vector $\hat{\mathbf{y}}$ is called the **orthogonal projection of $\mathbf{y}$ onto** W and is denoted by $\text{proj}_W \mathbf{y}$.

- $\hat{\mathbf{y}}$ is the closest vector in W to $\mathbf{y}$. (This is actually Theorem 9, p. 398, in simpler language.)

- The distance from $\mathbf{y}$ to the subspace W is $||\mathbf{y} - \hat{\mathbf{y}}||$

Exercise

8. Find the closest point to $\mathbf{y}$ in $span\{\mathbf{u}_1, \mathbf{u}_2\}$, where

$$\mathbf{y} = \begin{bmatrix} 2 \\ 4 \\ 0 \\ -2 \end{bmatrix}, \quad \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

Solution: The closest point to $\mathbf{y}$ in $span\{\mathbf{u}_1, \mathbf{u}_2\}$ is the sum of the components of $\mathbf{y}$ along $\mathbf{u}_1$ and $\mathbf{u}_2$. Thus:

$$\text{closest point } = \hat{\mathbf{y}} = \left(\frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\right) \mathbf{u}_1 + \left(\frac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\right) \mathbf{u}_2$$

$$=$$

• Orthonormal sets, Orthogonal matrices

Definitions A set $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is an **orthonormal set** if it is an orthogonal set and $||\mathbf{u}_i|| = 1$ for $i = 1 \ldots p$. If W is the subspace spanned by such a set, then $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is an **orthonormal basis** for W.

Example Show that

$$\left\{ \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \right\} = \{\mathbf{u}_1, \mathbf{u}_2\}$$

is an orthonormal set.

Solution:

- $\mathbf{u}_1 \cdot \mathbf{u}_2 = 0$ so the vectors are OG. Moreover,
- $||\mathbf{u}_1|| = \sqrt{\mathbf{u}_1 \cdot \mathbf{u}_1} = \sqrt{(2/3)^2 + (-2/3)^2 + (1/3)^2} = \sqrt{4/9 + 4/9 + 1/9} = 1$, and
- $||\mathbf{u}_2|| = \sqrt{(1/\sqrt{2})^2 + (1/\sqrt{2})^2 + 0^2} = \sqrt{1/2 + 1/2} = 1.$

So $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthonormal basis for the subspace (a plane) of $\mathbb{R}^3$ spanned by this set.

Observe: Suppose $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthonormal set in $\mathbb{R}^2$. Let U be the matrix $[\ \mathbf{u}_1\ \ \mathbf{u}_2\ ]$. Then

$$U^TU = \left[\begin{array}{c} \mathbf{u}_1^T \\ \hline \mathbf{u}_2^T \end{array}\right]\left[\begin{array}{c|c} \mathbf{u}_1 & \mathbf{u}_2 \end{array}\right] = \begin{bmatrix} \mathbf{u}_1^T\mathbf{u}_1 & \mathbf{u}_1^T\mathbf{u}_2 \\ \mathbf{u}_2^T\mathbf{u}_1 & \mathbf{u}_2^T\mathbf{u}_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Theorem 6 An $m \times n$ matrix U has orthonormal columns if and only if $U^TU = I_n$.

Example Recall that $\left\{\begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}\right\}$ is an orthonormal set. Note that

$$\begin{bmatrix} 2/3 & -2/3 & 1/3 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}\begin{bmatrix} 2/3 & 1/\sqrt{2} \\ -2/3 & 1/\sqrt{2} \\ 1/3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Other theorems about orthogonal matrices:

Theorem 7 Let U be an $m \times n$ matrix with orthonormal columns, and let $\mathbf{x}$ and $\mathbf{y}$ be in $\mathbb{R}^n$. Then

a. $||U\mathbf{x}|| = ||\mathbf{x}||$

b. $(U\mathbf{x}) \cdot (U\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$

c. $(U\mathbf{x}) \cdot (U\mathbf{y}) = 0$ if and only if $\mathbf{x} \cdot \mathbf{y} = 0$

Theorem 10 If $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is an *orthonormal* basis for a subspace W of $\mathbb{R}^n$, then

$$\text{proj}_W \mathbf{y} = (\mathbf{y} \cdot \mathbf{u}_1)\mathbf{u}_1 + \cdots + (\mathbf{y} \cdot \mathbf{u}_p)\mathbf{u}_p.$$

Moreover, if $U = [\ \mathbf{u}_1\ \mathbf{u}_2\ \cdots\ \mathbf{u}_p\ ]$, then

$$\text{proj}_W \mathbf{y} = UU^T\mathbf{y} \quad \forall\ \mathbf{y} \in \mathbb{R}^n.$$

Exercises: see homework

Summary

- Orthogonality $\Rightarrow$ LI, but LI does not imply orthogonality
- $\mathrm{proj}_{\mathbf{u}}\mathbf{y} = \left(\frac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u}$ = component of $\mathbf{y}$ along $\mathbf{u}$.
- $\mathrm{proj}_{W}\mathbf{y} = \left(\frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\right)\mathbf{u}_1 + \cdots + \left(\frac{\mathbf{y}\cdot\mathbf{u}_p}{\mathbf{u}_p\cdot\mathbf{u}_p}\right)\mathbf{u}_p$, where $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is an *orthogonal* basis of W. [warning– this isn't true if the $\mathbf{u}_i$ are merely LI. They must be OG.]
- In the above expression, $\mathbf{u}_i \cdot \mathbf{u}_i = 1$ for $i = 1 \ldots p$ if the basis is *orthonormal*.
- Nearest point to $\mathbf{y}$ in W, distance from $\mathbf{y}$ to W
- Orthogonal matrices, $U^T U = I$