Section 6.1 – Inner Product, Length, Orthogonality

Main Ideas in this section:

- Inner Products of vectors, length/norm of a vector
- Orthogonal Vectors
- Orthogonal Complements of Vector Spaces

• **Inner Products:** Given two vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, the **inner product** (or **dot product**) of $\mathbf{v}$ and $\mathbf{w}$ is the *scalar* $\mathbf{v}^T\mathbf{w} = v_1w_1 + v_2w_2 + \cdots + v_nw_n$.

Theorem 1 Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors in $\mathbb{R}^n$, and let c be a scalar.

a. $\mathbf{v} \cdot \mathbf{u} = \mathbf{u} \cdot \mathbf{v}$

b. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = (\mathbf{u} \cdot \mathbf{w}) + (\mathbf{u} \cdot \mathbf{v})$

c. $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$

d. $\mathbf{u} \cdot \mathbf{u} \geq 0$, and $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$

$$\Rightarrow (c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p) \cdot \mathbf{w} = c_1(\mathbf{u}_1 \cdot \mathbf{w}) + \cdots + c_p(\mathbf{u}_p \cdot \mathbf{w})$$

Definition: The **length** (or **norm**) of $\mathbf{v}$ is the nonnegative scalar $||\mathbf{v}||$ defined as follows:

$$||\mathbf{v}|| = \sqrt{\mathbf{v}\cdot\mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}, \quad \text{and} \quad ||\mathbf{v}||^2 = \mathbf{v}\cdot\mathbf{v}$$

Exercises

1. Given $\mathbf{u} = (3, 4)$ and $\mathbf{v} = (5, 1)$, find $\mathbf{u}\cdot\mathbf{v}$, $||\mathbf{u}||$, $||\mathbf{v}||$, and find the distance between $\mathbf{u}$ and $\mathbf{v}$.
2. Given $\mathbf{u}$ above, find a vector $\mathbf{w}$ in the same direction as $\mathbf{u}$, but one for which $||\mathbf{w}|| = 1$.

   ⋆⋆ A vector of length 1 is called a **unit vector**.

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• **Orthogonal Vectors** Two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ are **orthogonal** to each other if $\mathbf{u}\cdot\mathbf{v} = 0$.

Exercise Determine which pair(s) of vectors are orthogonal.

$(4, 3),\ (-3, 4)$ $\quad$ $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right),\ \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ $\quad$ $(10, 2),\ (1, -5)$

⋆⋆ Another word for "orthogonal" is **perpendicular**.

Exercises

1. Show that if vector $\mathbf{y}$ is orthogonal to vectors $\mathbf{u}_1$ and $\mathbf{u}_2$, then $\mathbf{y}$ is orthogonal to $\mathbf{u}_1 + \mathbf{u}_2$.

2. Sketch three vectors in $\mathbb{R}^3$ that demonstrate the result of problem 1.

• Orthogonal Complements

Consider $\mathbb{R}^3$, and let W be the subspace $\left\{ \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} : \ x_1, x_2 \in \mathbb{R} \right\}$. Let $\mathbf{z}$ be the vector $(0, 0, 1) \in \mathbb{R}^3$. Then $\mathbf{z}$ is orthogonal to *every* vector in W.

In fact,

- every vector in span$\{\mathbf{z}\}$ is OG to every vector in W, and
- no vector in $\mathbb{R}^3$ *outside* of span$\{\mathbf{z}\}$ is OG to vectors in W.

Definition The set of all vectors that are orthogonal to a subspace W of $\mathbb{R}^n$ is called the **orthogonal complement** of W, and is denoted $W^\perp$.

Examples/Remarks

- Let the subspace W of $\mathbb{R}^2$ be the line that passes through $(0,0)$ with slope 1. Then $W^\perp$ is the line that passes through $(0,0)$ with slope -1.
- Let W be the x_1 axis in $\mathbb{R}^3$. Then $W^\perp$ is the x_2x_3 plane.
- Given a subspace W of $\mathbb{R}^n$, a vector $\mathbf{x}$ is in $W^\perp$ if and only if $\mathbf{x}$ is orthogonal to every vector in a set that spans W.
- Given a subspace W of $\mathbb{R}^n$, $W^\perp$ is also a subspace of $\mathbb{R}^n$.

Summary

- $\mathbf{u} \cdot \mathbf{v}$ is a *number*
- $||\mathbf{u}||$ is the length of vector $\mathbf{u}$, computed as $\sqrt{\mathbf{u} \cdot \mathbf{u}}$
- $\mathbf{u}, \mathbf{v}$ are orthogonal $\Leftrightarrow \mathbf{u} \cdot \mathbf{v} = 0$.
- Orthogonal complement $W^\perp$ of a subspace W of $\mathbb{R}^n$ is the set of all $\mathbf{v} \in \mathbb{R}^n$ that are orthogonal to *all* $\mathbf{w} \in W$. That is, $W^\perp = \{\mathbf{v} \in \mathbb{R}^n : \mathbf{v} \cdot \mathbf{w} = 0 \ \forall \ \mathbf{w} \in W\}$.