Section 6.1 – Inner Product, Length, Orthogonality

Main Ideas in this section:

- Inner Products of vectors, length/norm of a vector
- Orthogonal Vectors
- Orthogonal Complements of Vector Spaces

• Inner Products: Given two vectors \( \mathbf{v}, \mathbf{w} \in \mathbb{R}^n \), the inner product (or dot product) of \( \mathbf{v} \) and \( \mathbf{w} \) is the scalar \( \mathbf{v}^T \mathbf{w} = v_1w_1 + v_2w_2 + \cdots + v_nw_n \).

Theorem 1  Let \( \mathbf{u}, \mathbf{v}, \mathbf{w} \) be vectors in \( \mathbb{R}^n \), and let \( c \) be a scalar.

a. \( \mathbf{v} \cdot \mathbf{u} = \mathbf{u} \cdot \mathbf{v} \)
b. \( (\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = (\mathbf{u} \cdot \mathbf{w}) + (\mathbf{u} \cdot \mathbf{v}) \)
c. \( (c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v}) \)
d. \( \mathbf{u} \cdot \mathbf{u} \geq 0, \) and \( \mathbf{u} \cdot \mathbf{u} = 0 \) if and only if \( \mathbf{u} = \mathbf{0} \)

\[ \Rightarrow (c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p) \cdot \mathbf{w} = c_1(\mathbf{u}_1 \cdot \mathbf{w}) + \cdots + c_p(\mathbf{u}_p \cdot \mathbf{w}) \]
Definition:  The length (or norm) of \( \mathbf{v} \) is the nonnegative scalar \( ||\mathbf{v}|| \) defined as follows:

\[
||\mathbf{v}|| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}, \quad \text{and} \quad ||\mathbf{v}||^2 = \mathbf{v} \cdot \mathbf{v}
\]

Exercises

1. Given \( \mathbf{u} = (3, 4) \) and \( \mathbf{v} = (5, 1) \), find \( \mathbf{u} \cdot \mathbf{v} \), \( ||\mathbf{u}|| \), \( ||\mathbf{v}|| \), and find the distance between \( \mathbf{u} \) and \( \mathbf{v} \).

2. Given \( \mathbf{u} \) above, find a vector \( \mathbf{w} \) in the same direction as \( \mathbf{u} \), but one for which \( ||\mathbf{w}|| = 1 \).

\( \star \star \) A vector of length 1 is called a unit vector.

- **Orthogonal Vectors**  Two vectors \( \mathbf{u} \) and \( \mathbf{v} \) in \( \mathbb{R}^n \) are orthogonal to each other if \( \mathbf{u} \cdot \mathbf{v} = 0 \).

Exercise Determine which pair(s) of vectors are orthogonal.

\( (4, 3), \; (-3, 4) \quad \left( \frac{\sqrt{3}}{2}, \frac{1}{2} \right), \; \left( \frac{\sqrt{3}}{2}, -\frac{1}{2} \right) \quad (10, 2), \; (1, -5) \)

\( \star \star \) Another word for “orthogonal” is perpendicular.
Exercises

1. Show that if vector $\mathbf{y}$ is orthogonal to vectors $\mathbf{u}_1$ and $\mathbf{u}_2$, then $\mathbf{y}$ is orthogonal to $\mathbf{u}_1 + \mathbf{u}_2$.

2. Sketch three vectors in $\mathbb{R}^3$ that demonstrate the result of problem 1.

• Orthogonal Complements

Consider $\mathbb{R}^3$, and let $W$ be the subspace
\[
\left\{ \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} : x_1, x_2 \in \mathbb{R} \right\}. \]
Let $\mathbf{z}$ be the vector $(0, 0, 1) \in \mathbb{R}^3$. Then $\mathbf{z}$ is orthogonal to every vector in $W$.

In fact,
- every vector in $\text{span}\{\mathbf{z}\}$ is OG to every vector in $W$, and
- no vector in $\mathbb{R}^3$ outside of $\text{span}\{\mathbf{z}\}$ is OG to vectors in $W$. 


**Definition**  The set of all vectors that are orthogonal to a subspace $W$ of $\mathbb{R}^n$ is called the **orthogonal complement** of $W$, and is denoted $W^\perp$.

**Examples/Remarks**

- Let the subspace $W$ of $\mathbb{R}^2$ be the line that passes through $(0, 0)$ with slope $1$. Then $W^\perp$ is the line that passes through $(0, 0)$ with slope $-1$.

- Let $W$ be the $x_1$ axis in $\mathbb{R}^3$. Then $W^\perp$ is the $x_2x_3$ plane.

- Given a subspace $W$ of $\mathbb{R}^n$, a vector $\mathbf{x}$ is in $W^\perp$ if and only if $\mathbf{x}$ is orthogonal to every vector in a set that spans $W$.

- Given a subspace $W$ of $\mathbb{R}^n$, $W^\perp$ is also a subspace of $\mathbb{R}^n$.

**Summary**

- $\mathbf{u} \cdot \mathbf{v}$ is a **number**

- $||\mathbf{u}||$ is the length of vector $\mathbf{u}$, computed as $\sqrt{\mathbf{u} \cdot \mathbf{u}}$

- $\mathbf{u}, \mathbf{v}$ are orthogonal $\iff \mathbf{u} \cdot \mathbf{v} = 0$.

- Orthogonal complement $W^\perp$ of a subspace $W$ of $\mathbb{R}^n$ is the set of all $\mathbf{v} \in \mathbb{R}^n$ that are orthogonal to all $\mathbf{w} \in W$. That is, $W^\perp = \{ \mathbf{v} \in \mathbb{R}^n : \mathbf{v} \cdot \mathbf{w} = 0 \ \forall \ \mathbf{w} \in W \}$. 