Section 5.5 – Complex Eigenvalues

Main Ideas in this section:

- $A\mathbf{x} = \lambda\mathbf{x}$ when $\lambda \in \mathbb{C}$ and $\mathbf{x} \in \mathbb{C}^n$
- Rotation and scaling in the transformation $\mathbf{v} \mapsto A\mathbf{v}$ when eigenvalues ($\mathbf{x}$) and eigenvectors (λ) of A are complex.

Conceptual background

- $A\mathbf{x} = \lambda\mathbf{x}$ when $\lambda \in \mathbb{C}$ and $\mathbf{x} \in \mathbb{C}^n$

Consider $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

- The transformation $\mathbf{x} \mapsto A\mathbf{x}$ rotates points by $\pi/2$.

- No vector in $\mathbb{R}^2$ is mapped to a scalar multiple of itself

  $\Rightarrow$ A has no eigenvectors in $\mathbb{R}^2$.

- But A has a characteristic polynomial $\lambda^2 + 1$ of degree 2, ... hence 2 eigenvalues:

Exercises

1. For $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, eigenvalues are $\lambda = \pm i$. Find the eigenspace corresponding to $\lambda = i$.
2. Let $A = \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix}$. Find the eigenvalues of A, and find a basis for each eigenspace.

• Rotation and scaling

Background:

- Let $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, with a and b real and nonzero.
- Eigenvalues of C are $\lambda = a \pm bi$.
- Let $r = |\lambda| = \sqrt{a^2 + b^2}$, and let φ be the angle between the positive x-axis and the line through $(0, 0)$ and (a, b).
- Note that $C = r \begin{bmatrix} a/r & -b/r \\ b/r & a/r \end{bmatrix} = r \begin{bmatrix} cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$.

$\star$ Thus, for any $\mathbf{v} \in \mathbb{C}^2$, $\mathbf{v} \mapsto C\mathbf{v}$ is the composition of a rotation through the angle φ and a scaling by $|\lambda|$.

Examples

1. Let $A = \begin{bmatrix} 0.8746 & -0.4848 \\ 0.4848 & 0.8746 \end{bmatrix}$.
   - Eigenvalues of A are:
   - Eigenvectors are *not* real, can't be graphed in $\mathbb{R}^2$.
   - $r = |\lambda| =$
   - Angle of rotation: $\varphi = \cos^{-1}(.8746) = +29^o$
   - Observe the behavior of iterated transformations of a vector in $\mathbb{R}^2$...

2. Let $A = \begin{bmatrix} 2 & 2 \\ -0.55 & 0 \end{bmatrix}$.

- Eigenvalues of A are:
- Eigenvectors are *not* real, can't be graphed in $\mathbb{R}^2$.
- $|\lambda| = 1.0488$
- Angle of rotation ... undetermined.
- Observe the behavior of iterated transformations of a vector in $\mathbb{R}^2$...

Observe the rotation that's hidden inside a matrix similar to A:

3. Let $A = \begin{bmatrix} 2 & 2 \\ -0.55 & 0 \end{bmatrix}$.

- Eigenvalues of A are $\lambda = 1 \pm i\sqrt{0.1}$, $|\lambda| = 1.0488$.
- An eigenvector corresponding to $\lambda = 1 + i\sqrt{0.1}$ is $\mathbf{x} = \begin{bmatrix} -2 \\ 1 - i\sqrt{0.1} \end{bmatrix}$.
- Let $P = [\mathrm{Re}(\mathbf{x}) \;\; \mathrm{Im}(\mathbf{x})] = \begin{bmatrix} -2 & 0 \\ 1 & -\sqrt{0.1} \end{bmatrix}$
- Then $P^{-1}AP$

$$= \begin{bmatrix} -2 & 0 \\ 1 & -\sqrt{0.1} \end{bmatrix}^{-1} \begin{bmatrix} 2 & 2 \\ -0.55 & 0 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 1 & -\sqrt{0.1} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & .31 \\ -.31 & 1 \end{bmatrix}$$

$$= 1.0488 \begin{bmatrix} 1/1.0488 & .31/1.0488 \\ -.31/1.0488 & 1/1.0488 \end{bmatrix}$$

$$= 1.0488 \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}, \text{ where } \varphi \approx -17.5^o.$$

Summary

- Finding complex λ's is no different from finding real λ's.
- If A has real entries, and if $(\lambda,\ \mathbf{x})$ are a complex eigenvalue-eigenvector pair, then $(\bar{\lambda},\ \bar{\mathbf{x}})$ form another eigen-pair for A.
- Finding complex eigenvectors is simplified by knowing that $A - \lambda I$ has nontrivial solutions, hence linearly dependent rows.
- Rotations of $\mathbf{v}$ when $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, or when A is similar to such a matrix, can be described by an angle φ and a scaling factor $r = |\lambda|$.