Section 5.4 – Eigenvectors and Linear Transformations

Main Ideas in this section:

- Linear Transform $T : V \to W$, with $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ a basis for V, $\mathcal{D} = \{\mathbf{d}_1, \ldots, \mathbf{d}_m\}$ a basis for W.
- Standard matrix of $[\mathbf{v}]_{\mathcal{B}} \mapsto [T(\mathbf{v})]_{\mathcal{D}}$ for $\mathbf{v} \in V$
- Matrix Algebra

• **Linear Transformations** $T : V \to W$, bases $\mathcal{B}$ and $\mathcal{D}$, **Conceptual Underpinnings:**

⊳⊳ Let V and W be vector spaces with $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ a basis for V and $\mathcal{D} = \{\mathbf{d}_1, \ldots, \mathbf{d}_m\}$ a basis for W, and let $T : V \to W$ be a linear transformation. Take $\mathbf{v} = c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n \in V$, and suppose $T(\mathbf{v}) = t_1\mathbf{d}_1 + \ldots + t_m\mathbf{d}_m \in W$.

⊳⊳ $\mathbf{v}$ and $T(\mathbf{v})$ may not be elements of $\mathbb{R}^k$ or $\mathbb{R}^l$ for any k or l, but one thing is certain: $[\mathbf{v}]_{\mathcal{B}} \in \mathbb{R}^n$ and $[T(\mathbf{v})]_{\mathcal{D}} \in \mathbb{R}^m$.

⊳⊳ Since $[\mathbf{v}]_{\mathcal{B}} \mapsto [T(\mathbf{v})]_{\mathcal{D}}$ is a transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$, there *should* be a standard matrix for it.

- **Standard matrix of** $[\mathbf{v}]_{\mathcal{B}} \mapsto [T(\mathbf{v})]_{\mathcal{D}}$ for $\mathbf{v} \in V$

1. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{D} = \{\mathbf{d}_1, \mathbf{d}_2\}$ be bases for vector spaces V and W, respectively, and let $T : V \to W$ be a linear transformation with
$$T(\mathbf{b}_1) = -3\mathbf{d}_1 + 4\mathbf{d}_2 \quad \text{and} \quad T(\mathbf{b}_2) = \mathbf{d}_1 - 2\mathbf{d}_2.$$
Find the standard matrix M of the transformation
$$[\mathbf{v}]_{\mathcal{B}} \mapsto [T(\mathbf{v})]_{\mathcal{D}}.$$
⋆⋆ This matrix is called the **standard matrix of T relative to $\mathcal{B}$ and $\mathcal{D}$.**

2. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4\}$ be a basis for vector space V and let $T : V \to \mathbb{R}^3$ be a linear transformation with the property that
$$T(c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + c_3\mathbf{b}_3 + c_4\mathbf{b}_4) = \begin{bmatrix} 3c_2 + 4c_3 - c_4 \\ c_1 - 7c_3 \\ 2c_1 + 5c_2 \end{bmatrix}.$$
Find $[T(\mathbf{b}_1)]_{\mathcal{E}}, [T(\mathbf{b}_2)]_{\mathcal{E}}, [T(\mathbf{b}_3)]_{\mathcal{E}}, [T(\mathbf{b}_4)]_{\mathcal{E}}$, then find the matrix of T relative to the bases of V and $\mathbb{R}^3$.

3. Let $T : \mathbb{P}_2 \to \mathbb{P}_3$ be the transformation that maps a polynomial $\mathbf{p}(t)$ to $3t\mathbf{p}(t)$.
   (a) Find the image of $\mathbf{p}(t) = 5 + 3t - t^2$.
   (b) What is the range of T?
   (c) Show that T is a linear transformation.
   (d) Find the matrix of T relative to the standard bases for $\mathbb{P}_2$ and $\mathbb{P}_3$.

Short version: Given vector spaces V and W, a linear transformation $T : V \to W$, bases $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ and $\mathcal{D}$ of V, W, respectively. The matrix of T relative to $\mathcal{B}$ and $\mathcal{D}$ is

$$M = [\ [T(\mathbf{b}_1)]_{\mathcal{D}} \ \ [T(\mathbf{b}_2)]_{\mathcal{D}} \ \ \cdots \ \ [T(\mathbf{b}_n)]_{\mathcal{D}}\ ].$$

Using this matrix, we have

$$[T(\mathbf{v})]_{\mathcal{D}} = M[\mathbf{v}]_{\mathcal{B}}.$$

Exercises

1. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be a basis for V, let $\mathcal{T} = \{2 + t, 1 - t\}$ be a basis for $\mathbb{P}_1$, and let T be the linear transform $T : V \to \mathbb{P}_1$ whose matrix relative to $\mathcal{B}$ and $\mathcal{T}$ is
$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 5 \end{bmatrix}.$$
Find $T(2\mathbf{b}_1 + 3\mathbf{b}_2 - \mathbf{b}_3)$.

2. Let $T : \mathbb{P}_2 \to \mathbb{R}^3$ be given by $T(\mathbf{p}(t)) = \begin{bmatrix} \mathbf{p}(1) \\ \mathbf{p}'(2) \\ \mathbf{p}''(0) \end{bmatrix}$.

   (a) Find $T(\mathbf{p})$ for $\mathbf{p}(t) = 3 - 5t + 3t^2$.

   (b) Find the matrix of T relative to the standard bases for $\mathbb{P}_2$ and $\mathbb{R}^3$. Use this matrix to verify your answer to part (a).

   (c) Find $\mathbf{p}(t)$ for which $T(\mathbf{p}(t)) = \begin{bmatrix} 4 \\ 3 \\ 4 \end{bmatrix}$.

- **Matrix Algebra**

1. Show that if A is similar to B, then A^2 is similar to B^2.
2. Show that if A is diagonalizable and A is similar to B, then B is diagonalizable.
3. Add 19, 21, 23 to your homework for 5.4.

Summary

- With bases $\mathcal{B}$ and $\mathcal{D}$ of *any* vector spaces V and W, along with the linear transformation $T : V \to W$, we can find a standard matrix of T relative to $\mathcal{B}$ and $\mathcal{D}$.
- "Global" properties allow you to prove linearity, similarity, etc. without getting into minute details.