Section 5.3 – Diagonalization

Main Ideas in this section:

- Computing A^k quickly if A is similar to a diagonal matrix
- Diagonalizable matrices
- Connections to Eigenvectors

- Computing A^k quickly if A is similar to a diagonal matrix

1. Use the fact that
$$A = \begin{bmatrix} -3 & 12 \\ -2 & 7 \end{bmatrix} = \underbrace{\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}^{-1}}_{P^{-1}}$$ to compute A^3.
2. If $A = PDP^{-1}$ for some invertible P and diagonal D, with all matrices size $n \times n$, find a formula to compute A^k without computing $\underbrace{AA\cdots A}_{k\ factors}$.

- Diagonalizable matrices (definition): A square matrix A is **diagonalizable** if A is similar to a diagonal matrix.

- Connections to eigenvectors:

Theorem 5: The Diagonalization Theorem

- An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.
- In fact, $A = PDP^{-1}$, with D a diagonal matrix, if and only if the columns of P are n linearly independent eigenvectors of A. In this case, the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors of P.

Partial proof:

1. Diagonalize A, if possible. That is, find P and D such that $A = PDP^{-1}$, for $A = \begin{bmatrix} 0 & -4 & -6 \\ -1 & 0 & -3 \\ 1 & 2 & 5 \end{bmatrix}$.

   Note: $\lambda = 2, 2, 1$.

2. Diagonalize A, if possible, for $A = \begin{bmatrix} -3 & 1 \\ 0 & -3 \end{bmatrix}$.

3. Diagonalize A, if possible, for $A = \begin{bmatrix} 2 & 1 & 4 \\ 0 & 1 & -3 \\ 0 & 0 & 2 \end{bmatrix}$.

Lessons to glean from the above examples:

- It's not necessary for an $n \times n$ matrix to have n *distinct* eigenvalues in order to be diagonalizable. WHAT MATTERS is linear independence of eigenvectors.

- Two matrices with the same eigenvalues, with the same multiplicities, aren't necessarily *both* diagonalizable, or both *not* diagonalizable. WHAT MATTERS is linear independence of eigenvectors.

$\star\star\star$ In other words, A $(n \times n)$ is diagonalizable if and only if there are enough linearly independent eigenvectors to form a basis of $\mathbb{R}^n$. Such a basis is called an **eigenvector basis**. We use these eigenvectors to construct P, with corresponding eigenvalues for D, in order to diagonalize A.

Theorem 6 An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

Theorem 7 Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \ldots, \lambda_p$.

a. For $1 \leq k \leq p$, the dimension of the eigenspace for λ_k is less than or equal to the multiplicity of the eigenvalue λ_k.

b. The matrix A is diagonalizable if and only if the sum of the dimensions of the distinct eigenspaces equals n, and this happens if and only if the dimension of the eigenspace for each λ_k equals the multiplicity of λ_k.

c. If A is diagonalizable and β_k is a basis for the eigenspace corresponding to λ_k for each k, then the total collection of vectors in the sets $\beta_1, \ldots, \beta_p$ forms an eigenvector basis for $\mathbb{R}^n$.

Further exercises:

1. A is a 5×5 matrix with two eigenvalues. Each eigenspace is two-dimensional. Is A diagonalizable?
2. The eigenvalues of a 4×4 matrix A are $\lambda = 0, 0, 2, -3$. Is A similar to a diagonal matrix?
3. A is a 6×5 matrix with five pivot columns. Is A diagonalizable?
4. Construct a matrix that is diagonalizable but not invertible.
5. Construct a matrix that is invertible but not diagonalizable.

Summary

- A^k easy to compute if A is diagonalizable
- A $(n \times n)$ is diagonalizable $\Leftrightarrow$ A has a full set (n) of LI eigenvectors ("eigenvector basis")$\Leftrightarrow$ sum of dimensions of *distinct* eigenspaces $= n$.
- Build P from eigenvector basis, D from eigenvalues