Section 5.1 – Eigenvectors and Eigenvalues

Main Ideas in this section:

- Definitions and Concepts
- Finding Eigenvectors, Eigenspaces

The goal in this chapter is to dissect the action of a linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ into elements that are easily visualized. (Note: through most of chapter 5, A is $n \times n$.)

Conceptual Framework:

- Consider any vector $\mathbf{x}$ in $\mathbb{R}^2$. Using the standard basis, we can express $\mathbf{x}$ as $\mathbf{x} = c_1\mathbf{e}_1 + c_2\mathbf{e}_2$ for a unique set of scalars c_1, c_2.
- We know that a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is completely described in terms of its action on basis vectors.
- Geometry of a linear transformation:

What we seek: basis vectors that stretch (dilate) or shrink (contract), but don't rotate, upon transformation by T.
Geometry:

Definition An **eigenvector** of an $n \times n$ matrix A is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. The scalar λ is called the **eigenvalue** associated with eigenvector $\mathbf{x}$.

Comments:

- Note the geometry: if $\mathbf{x}$ is an eigenvector of A, then $A \cdot \mathbf{x}$ serves to dilate or contract $\mathbf{x}$, but does not rotate it.

- If eigenvector $\mathbf{x}$ has corresponding eigenvalue λ, then ANY $\mathbf{v} \in \text{Span}(\{\mathbf{x}\})$ is an eigenvector with eigenvalue λ.

- Eigenvector $\mathbf{x} \neq \mathbf{0}$, but eigenvalue λ can be 0.

- Only square matrices have eigenvectors and eigenvalues.

Exercise Determine whether or not $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ are eigenvectors of $A = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix}$.

Exercise: Given λ, find $\mathbf{x}$.

Given that 4 is an eigenvalue of $A = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix}$, find the corresponding eigenvectors.

The above method works for any eigenvalue λ of A. That is, given λ, we find corresponding eigenvectors by solving the homogeneous equation $(A - \lambda I)\mathbf{x} = \mathbf{0}$. Since the solutions (i.e. the null space of the $n \times n$ matrix $(A - \lambda I)$) comprise a subspace of $\mathbb{R}^n$, the set of all eigenvectors corresponding to λ is a subspace of $\mathbb{R}^n$ and is called the **eigenspace** of A corresponding to λ.

Exercise Find a basis for the eigenspace of
$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$ corresponding to $\lambda = 2$.

Question: Geometrically, what happens to *any* vector $\mathbf{v}$ in the eigenspace (plane in $\mathbb{R}^3$) from the example above when $\mathbf{v}$ is multiplied by A?

Exercise Suppose λ is an eigenvalue of A. Determine an eigenvalue of A^2, A^3, and A^n for any positive integer n.

Exercise A particular matrix A has eigenvalue $\lambda = 0$. Describe the corresponding eigenspace. Can anything be said about A? [Hint: IMT]

Theorem 1 The eigenvalues of a triangular matrix are the entries on its main diagonal.

Theorem 2 If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is linearly independent.

Summary

- Eigenvalues and Eigenvectors, $A\mathbf{x} = \lambda\mathbf{x}$
- Eigenspace corresponding to λ is "invariant" under multiplication by A.
- The eigenspace of A corresp. to λ equals Null$(A - \lambda I)$.
- Eigenvectors corresponding to distinct eigenvalues are linearly independent.
- $\lambda = 0 \iff A$ has nontrivial null space $\iff \det A = 0 \iff A$ is singular $\iff \ldots$