Section 4.4 – Coordinate Systems

Main Ideas in this section:

- Coordinate mappings - Imposing coordinate systems on Vector Spaces
- Isomorphisms
- Change of coordinates between different bases of $\mathbb{R}^n$

In general, people are more comfortable working with $\mathbb{R}^n$ and its subspaces than with other types of vector spaces and subspaces. The goal in this section is to define a correspondence between the $\mathbb{R}^n$ coordinate system and other vector spaces, even if they are not $\mathbb{R}^n$.

Theorem 7 The Unique Representation Theorem

Let $\beta = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Then $\forall\ \mathbf{x} \in V,\ \exists!$ set of scalars $c_1, \ldots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n.$$

Comments:

- It's the *linear independence* of basis vectors that makes the set of scalars unique
- Compare this to the IMT– unique solution $\forall\ \mathbf{b} \in$ range(A).

Example

Let $\beta = \{1, t, t^2\}$ be a basis for $\mathbb{P}_2$. Then for $\mathbf{p}(t) = 2 - 3t^2$, the unique set of scalars c_1, c_2, c_3 is $(c_1, c_2, c_3) = (2, 0, -3)$.

Exercise

Let $\beta = \{1 - t, 1 + t, 1 + t + t^2\}$ be a basis for $\mathbb{P}_2$. Find the unique set of scalars (c_1, c_2, c_3) that represent $\mathbf{p}(t) = 2 - 3t^2$.

Definition. Suppose $\beta = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ is a basis for a vector space V and $\mathbf{x} \in V$. The **coordinates of $\mathbf{x}$ relative to the basis β**, or the **β-coordinates of $\mathbf{x}$**, are the weights $c_1, \ldots, c_n$ such that $\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$.

In this case, the vector

$$[\mathbf{x}]_\beta = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

is called the **coordinate vector of $\mathbf{x}$ relative to β**, or the **β-coordinate vector of $\mathbf{x}$**

Example The β-coordinate vector for $\mathbf{p}(t)$ in the last exercise was

$$[2 - 3t^2]_\beta = \begin{bmatrix} \\ \\ \\ \end{bmatrix}.$$

Exercise Let $\mathbf{b}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} 0 & 2 \\ 2 & 0 \\ 0 & 1 \end{bmatrix}$, and define $V = \mathrm{Span}(\mathbf{b}_1, \mathbf{b}_2)$. Then $\{\mathbf{b}_1, \mathbf{b}_2\}$ is a basis for V. Call this basis β. Find the β-coordinate vector for

$$\mathbf{x} = \begin{bmatrix} 4 & 18 \\ 18 & 4 \\ 4 & 9 \end{bmatrix}.$$

Exercise Use $\mathbf{x}$ from the example above, and let β be the standard basis for $M_{3\times 2}$. That is, the ordered basis is $\beta = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$. Find $[\mathbf{x}]_\beta$.

Coordinate Mappings, Isomorphisms

Choosing an ordered basis $\beta = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ for a vector space V introduces a coordinate system in V. That is, the coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_\beta$ connects the possibly unfamiliar space V with the familiar space $\mathbb{R}^n$. Points in V can now be identified by their new "names" in $\mathbb{R}^n$, and every vector-space calculation in V is accurately reproduced in $\mathbb{R}^n$ (and vice versa.)

Example The parallel universes of $\mathbb{P}_2$ and $\mathbb{R}^3$.

$\mathbb{P}_2$	$\mathbb{R}^3$
$\mathbf{p}(t) = a + bt + ct^2$	$\begin{bmatrix} a \\ b \\ c \end{bmatrix}$
$(-1 + 2t - 3t^2) + (2 + 3t + 5t^2)$ $= 1 + 5t + 2t^2$	$\begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}$
$4(5t - 6t^2) = 20t - 24t^2$	$4\begin{bmatrix} 0 \\ 5 \\ -6 \end{bmatrix} = \begin{bmatrix} 0 \\ 20 \\ -24 \end{bmatrix}$

Theorem 8
Let $\beta = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Then the coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_\beta$ is a one-to-one linear transformation from V onto $\mathbb{R}^n$.

Proof:

Remarks:

- A *one-to-one* linear transformation from a vector space V *onto* a vector space W is called an **isomorphism**. Two isomorphic spaces (that is, two spaces linked by a one-to-one & onto linear transformation) may look like entirely different spaces, but as vector spaces they are indistinguishable!! This is evidenced by the fact that every vector space calculation in one space is accurately reproduced in the other, as in the parallel universes of $\mathbf{P}_2$ and $\mathbb{R}^3$.

- Linearity of the coordinate mapping means that

$$[c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p]_\beta = c_1[\mathbf{u}_1]_\beta + \cdots + c_p[\mathbf{u}_p]_\beta.$$

Change of Bases in $\mathbb{R}^n$

Exercise Let $\beta = \{\mathbf{b}_1, \mathbf{b}_2\}$, where $\mathbf{b}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and let $E = \{\mathbf{e}_1, \mathbf{e}_2\}$, where $\mathbf{e}_1$ and $\mathbf{e}_2$ are the standard unit vectors in $\mathbb{R}^2$. If $[\mathbf{x}]_\beta = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, find $[\mathbf{x}]_E$.

Transformations from one basis to another in $\mathbb{R}^n$

From the last example, where $V = \mathbb{R}^2$, note that

$$\begin{bmatrix} 6 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix},$$

or

$$[\mathbf{x}]_E = \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix} [\mathbf{x}]_\beta.$$

For a basis $\beta = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ for $\mathbb{R}^n$, define the matrix

$$P_\beta = [\ \mathbf{b}_1 \ \ \mathbf{b}_2 \ \ \cdots \ \ \mathbf{b}_n\ ] \text{ and suppose } [\mathbf{x}]_\beta = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

Then

$$\mathbf{x} = P_\beta [\mathbf{x}]_\beta.$$

We call P_β the **change of coordinates matrix** from β to the standard basis in $\mathbb{R}^n$.

Since P_β is a square matrix with linearly independent columns, P_β^{-1} exists by the IMT, so we can conclude that

$$[\mathbf{x}]_\beta = P_\beta^{-1}\mathbf{x},$$

therefore P_β^{-1} is a change of coordinates matrix from the standard basis in $\mathbb{R}^n$ to the basis β.

Exercise Let $\mathbf{b}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $\beta = \{\mathbf{b}_1, \mathbf{b}_2\}$.

(a.) Find the change of coordinates matrix P_β from β to the standard basis in $\mathbb{R}^2$, and find the change of coordinates matrix P_β^{-1} from the standard basis in $\mathbb{R}^2$ to β.

(b.) If $\mathbf{x} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$, find $[\mathbf{x}]_\beta$.

Example Let $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$, $B = \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix}$, and $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in \mathbb{R}^3$. Note that $AB = BA = I$. Find the coordinate vector for $\mathbf{x}$ relative to the basis comprised of the columns of A.

Example Let $\beta = \{\mathbf{b}_1, \mathbf{b}_2\}$ where $\mathbf{b}_1 = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$, and let $H = \text{Span}\{\mathbf{b}_1, \mathbf{b}_2\}$. Find $[\mathbf{x}]_\beta$ if $\mathbf{x} = \begin{bmatrix} 9 \\ 13 \\ 15 \end{bmatrix}$.

Solution: Find c_1, c_2 such that $c_1 \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 13 \\ 15 \end{bmatrix}$. The corresponding augmented system is

$$\begin{bmatrix} 3 & 0 & 9 \\ 3 & 1 & 13 \\ 1 & 3 & 15 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore $c_1 = 3, c_2 = 4$, and so $[\mathbf{x}]_\beta = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

Note: $\begin{bmatrix} 9 \\ 13 \\ 15 \end{bmatrix} \in \mathbb{R}^3$ is associated with the vector $\begin{bmatrix} 3 \\ 4 \end{bmatrix} \in \mathbb{R}^2$.

Picture:

Summary

- Unique representation theorem
- Isomorphism btw. v-space V (with n basis vectors) and $\mathbb{R}^n$
- Change of coordinate matrices in $\mathbb{R}^n$