4.3 Linearly Independent Sets; Bases

Definition

A set of vectors \( \{v_1, v_2, \ldots, v_p\} \) in a vector space \( V \) is said to be **linearly independent** if the vector equation

\[
c_1v_1 + c_2v_2 + \cdots + c_pv_p = 0
\]

has only the trivial solution \( c_1 = 0, \ldots, c_p = 0 \).

The set \( \{v_1, v_2, \ldots, v_p\} \) is said to be **linearly dependent** if there exists weights \( c_1, \ldots, c_p \), not all \( 0 \), such that

\[
c_1v_1 + c_2v_2 + \cdots + c_pv_p = 0.
\]

The following results from Section 1.7 are still true for more general vectors spaces.

A set containing the zero vector is linearly dependent.

A set of two vectors is linearly dependent if and only if one is a multiple of the other.

A set containing the zero vector is linearly independent.
EXAMPLE: \( \left\{ \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 2 \\ 3 & 0 \end{bmatrix} \right\} \) is a linearly __________________ set.

EXAMPLE: \( \left\{ \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 3 & 6 \\ 9 & 11 \end{bmatrix} \right\} \) is a linearly ______________ set since \( \begin{bmatrix} 3 & 6 \\ 9 & 11 \end{bmatrix} \) is not a multiple of \( \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \).

**Theorem 4**

An indexed set \( \{v_1, v_2, \ldots, v_p\} \) of two or more vectors, with \( v_1 \neq 0 \), is linearly dependent if and only if some vector \( v_j \) \((j > 1)\) is a linear combination of the preceding vectors \( v_1, \ldots, v_{j-1} \).

**EXAMPLE:** Let \( \{p_1, p_2, p_3\} \) be a set of vectors in \( \mathbb{P}_2 \) where \( p_1(t) = t, p_2(t) = t^2, \) and \( p_3(t) = 4t + 2t^2 \). Is this a linearly dependent set?

**Solution:** Since \( p_3 = _____p_1 + _____p_2, \) \( \{p_1, p_2, p_3\} \) is a linearly __________________ set.
A Basis Set

Let $H$ be the plane illustrated below. Which of the following are valid descriptions of $H$?

(a) $H = \text{Span}\{v_1, v_2\}$    (b) $H = \text{Span}\{v_1, v_3\}$

(c) $H = \text{Span}\{v_2, v_3\}$    (d) $H = \text{Span}\{v_1, v_2, v_3\}$

A basis set is an “efficient” spanning set containing no unnecessary vectors. In this case, we would consider the linearly independent sets $\{v_1, v_2\}$ and $\{v_1, v_3\}$ to both be examples of basis sets or bases (plural for basis) for $H$.

**DEFINITION**

Let $H$ be a subspace of a vector space $V$. An indexed set of vectors $\beta = \{b_1, \ldots, b_p\}$ in $V$ is a basis for $H$ if

(i) $\beta$ is a linearly independent set, and

(ii) $H = \text{Span}\{b_1, \ldots, b_p\}$. 
EXAMPLE: Let \( \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \), \( \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \), \( \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \).

Show that \( \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \) is a basis for \( \mathbb{R}^3 \). The set \( \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \) is called a standard basis for \( \mathbb{R}^3 \).

Solutions: (Review the IMT, page 129) Let

\[
A = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\]

Since \( A \) has 3 pivots, the columns of \( A \) are linearly independent by the IMT and the columns of \( A \) are linearly independent by IMT. Therefore, \( \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \) is a basis for \( \mathbb{R}^3 \).

EXAMPLE: Let \( S = \{1, t, t^2, \ldots, t^n\} \). Show that \( S \) is a basis for \( \mathbb{P}_n \).

Solution: Any polynomial in \( \mathbb{P}_n \) is in span of \( S \). To show that \( S \) is linearly independent, assume \( c_0 \cdot 1 + c_1 \cdot t + \cdots + c_n \cdot t^n = 0 \)

Then \( c_0 = c_1 = \cdots = c_n = 0 \). Hence \( S \) is a basis for \( \mathbb{P}_n \).
EXAMPLE: Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$.

Is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ a basis for $\mathbb{R}^3$?

Solution: Again, let $A = [\mathbf{v}_1 \mathbf{v}_2 \mathbf{v}_3] = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$. Using row reduction,

$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 5 \end{bmatrix}$

and since there are 3 pivots, the columns of $A$ are linearly independent and they span $\mathbb{R}^3$ by the IMT. Therefore $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathbb{R}^3$. 
EXAMPLE: Explain why each of the following sets is not a basis for $\mathbb{R}^3$.

(a) \( \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} , \begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix} , \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} , \begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix} \right\} \)

(b) \( \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} , \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\} \)
Bases for Nul $A$

**EXAMPLE:** Find a basis for Nul $A$ where

$$A = \begin{bmatrix} 3 & 6 & 6 & 3 & 9 \\ 6 & 12 & 13 & 0 & 3 \end{bmatrix}.$$

**Solution:** Row reduce

$$\begin{bmatrix} A & 0 \end{bmatrix}:
\begin{bmatrix} 1 & 2 & 0 & 13 & 33 & 0 \\ 0 & 0 & 1 & -6 & -15 & 0 \end{bmatrix}
\quad x_1 = -2x_2 - 13x_4 - 33x_5
\quad x_3 = 6x_4 + 15x_5
\quad x_2, x_4 \text{ and } x_5 \text{ are free}

\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 - 13x_4 - 33x_5 \\ x_2 \\ 6x_4 + 15x_5 \\ x_4 \\ x_5 \end{bmatrix}

\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -13 \\ 0 \\ 6 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -33 \\ 0 \\ 15 \\ 0 \\ 1 \end{bmatrix}

\begin{bmatrix} \uparrow \\ u \\ \uparrow \\ v \end{bmatrix} + \begin{bmatrix} \uparrow \\ w \end{bmatrix}
Therefore \( \{u, v, w\} \) is a spanning set for \( \text{Nul } A \). In the last section we observed that this set is linearly independent. Therefore \( \{u, v, w\} \) is a basis for \( \text{Nul } A \). The technique used here always provides a linearly independent set.

**The Spanning Set Theorem**

A basis can be constructed from a spanning set of vectors by discarding vectors which are linear combinations of preceding vectors in the indexed set.

**EXAMPLE:** Suppose \( v_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \ v_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \) and 
\[ v_3 = \begin{bmatrix} -2 \\ -3 \end{bmatrix}. \]

**Solution:** If \( x \) is in \( \text{Span}\{v_1, v_2, v_3\} \), then
\[
    x = c_1 v_1 + c_2 v_2 + c_3 v_3 = c_1 v_1 + c_2 v_2 + c_3 (\_\_\_\_ v_1 + \_\_\_\_ v_2)
\]
\[
    = \_\_\_\_ v_1 + \_\_\_\_ v_2
\]
Therefore, 
\[ \text{Span}\{v_1, v_2, v_3\} = \text{Span}\{v_1, v_2\}. \]
THEOREM 5  The Spanning Set Theorem

Let \( S = \{v_1, \ldots, v_p\} \) be a set in \( V \) and let \( H = \text{Span}\{v_1, \ldots, v_p\} \).

a. If one of the vectors in \( S \) - say \( v_k \) - is a linear combination of the remaining vectors in \( S \), then the set formed from \( S \) by removing \( v_k \) still spans \( H \).

b. If \( H \neq \{0\} \), some subset of \( S \) is a basis for \( H \).
Bases for Col $A$

**EXAMPLE:** Find a basis for Col $A$, where

$$A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix}.$$  

**Solution:** Row reduce:

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} b_1 & b_2 & b_3 & b_4 \end{bmatrix}$$

Note that

$b_2 = \underline{\phantom{0}}b_1$ \quad and \quad $a_2 = \underline{\phantom{0}}a_1$

$b_4 = 4b_1 + 5b_3$ \quad and \quad $a_4 = 4a_1 + 5a_3$

$b_1$ and $b_3$ are not multiples of each other

$a_1$ and $a_3$ are not multiples of each other

Elementary row operations on a matrix do not affect the linear dependence relations among the columns of the matrix.

Therefore $\text{Span}\{a_1, a_2, a_3, a_4\} = \text{Span}\{a_1, a_3\}$ and $\{a_1, a_3\}$ is a basis for Col $A$.  

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THEOREM 6
The pivot columns of a matrix $A$ form a basis for $\text{Col} \ A$.

EXAMPLE: Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -2 \\ -4 \\ 6 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$.

Find a basis for $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

Solution: Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -4 & 6 \\ -3 & 6 & 9 \end{bmatrix}$ and note that $\text{Col} \ A = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

By row reduction, $A \sim \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Therefore a basis for $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} \right\}$. 
Review:

1. To find a basis for Nul $A$, use elementary row operations to transform $[A \ 0]$ to an equivalent reduced row echelon form $[B \ 0]$. Use the reduced row echelon form to find parametric form of the general solution to $Ax = 0$. The vectors found in this parametric form of the general solution form a basis for Nul $A$.

2. A basis for Col $A$ is formed from the pivot columns of $A$.
   **Warning:** Use the pivot columns of $A$, not the pivot columns of $B$, where $B$ is in reduced echelon form and is row equivalent to $A$. 
