Section 4.1 – Vector Spaces and Subspaces

Main Ideas in this section:

- Definition of a Vector Space
- Examples of Vector Spaces, old and new
- Subspaces
- New look at Span of a set of objects

A **Vector Space** is a set V of objects for which there are two defined operations, addition and scalar multiplication, subject to the ten rules below. The rules hold for all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in V and for all scalars c, d in $\mathbb{R}$.

1. $\mathbf{u} + \mathbf{v} \in V$
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
4. $\exists\ \mathbf{0} \in V$ such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$
5. $\forall\ \mathbf{u} \in V,\ \exists\ -\mathbf{u} \in V$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
6. $c\mathbf{u} \in V$
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$
10. $1\mathbf{u} = \mathbf{u}$

Most obvious example of a vector space: $\mathbb{R}^n$ for any integer n.

Other examples of vector spaces:

- Define $\mathbf{M}_{2\times 2}$ as

$$\mathbf{M}_{2\times 2} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}.$$

  $\mathbf{M}_{2\times 2}$ is a vector space.

- For any integer $n > 0$, define $\mathbb{P}_n$ as the set of all polynomials of degree at most n:

$$\mathbb{P}_n = \{p(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n\}$$

  where the coefficients $a_0, \ldots, a_n$ are real numbers.

  $\mathbb{P}_n$ is a vector space.

- Let V be the set of all real-valued continuous functions $\mathbf{f}(t)$ defined on the interval $[a, b]$. (Take note: This set is generally denoted by $\mathrm{C}[a, b]$.) If we define addition and scalar multiplication carefully*, then V is a vector space.

$\star$ carefully means: $\mathbf{f} + \mathbf{g}$ is the function whose value at t in the domain is $\mathbf{f}(t) + \mathbf{g}(t)$, and the function $c\mathbf{f}$ is the function whose value at t is $c\mathbf{f}(t)$.

TIP: try to think of an individual object in any vector space as a simple entity, for example as a point in space, even when the object is *not* defined so simply. $\star\star$ Most Important: every object in a vector space V obeys the ten rules for addition and scalar multiplication as defined for V.

Subspaces– special subsets of Vector Spaces

A **subspace** of a vector space V is a subset H of V that has three properties:

a. The zero vector of V is in H.

b. H is closed under vector addition. That is, for each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u}+\mathbf{v}$ is also in H.

c. H is closed under scalar multiplication. That is, for each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is also in H.

NOTE: Every subspace (not subset) of a vector space is itself a vector space.

Examples of subspaces:

- The set consisting of only the zero vector in any vector space V is a subspace of V. It is called the **zero subspace** and written as $\{\mathbf{0}\}$
- Let $H = span\left(\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}\right)$. Then H is a subspace of $\mathbb{R}^3$.

  Note that H looks and acts like $S = span\left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right)$. However, these are fundamentally different spaces. H is a subspace of $\mathbb{R}^3$, whereas S is not even a sub*set* of $\mathbb{R}^3$.

Exercise: Let H be the set of points inside and on the unit circle in the xy plane. That is, let $H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x^2 + y^2 \leq 1 \right\}$.

a. Is H a subset of $\mathbb{R}^2$?

b. Is H a subspace of $\mathbb{R}^2$?

Exercise: Let H be the set of all polynomials of the form $a + bt^2$, where a and b are in $\mathbb{R}$.

a. Is H a subset of $\mathbf{P}_2$?

b. Is H a subspace of $\mathbf{P}_2$? What if $b = 1$?

Theorem 1 If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ are objects in a vector space V, then $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is a subspace of V.

$\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is called the **subspace spanned** (or **generated**) by $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$. Given any subspace H of V, a **spanning** (or **generating**) **set** for H is a set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ such that $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$

Exercise: Let H be the set of all vectors of the form $(4a + 2b, a + b + c, 0, b - 2c)$. Is H a subspace of the vector space $\mathbb{R}^4$? If so, find its generating set.

Exercise: Let H be the set of all $\mathbf{x}$ in $\mathbb{R}^3$ such that $A\mathbf{x} = \mathbf{0}$, where $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix}$. Is H a subspace of $\mathbb{R}^3$? If so, find its spanning set.

Exercise: Let H and K be subspaces of a vector space V. The intersection of H and K, denoted $H \cap K$, is the set of $\mathbf{v}$ in V that belong to both H and K. Show that $H \cap K$ is a subspace of V.

Summary

- Vector Space = set of objects with rules for addition and scalar multiplication, and closure under these operations
- Objects in a vector space are not always vectors
- Checking subsets to see if they're subspaces– three criteria
- Span of a set of objects from V is automatically a subspace of V.