Section 2.3 – Characterizations of Invertible Matrices

Main Idea in this section:

- The Invertible Matrix Theorem– numerous equivalent statements about a nonsingular matrix A

Theorem 8 The Invertible Matrix Theorem
Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given $n \times n$ matrix A, the statements are either all true or all false.

a. A is an invertible matrix.

b. A is row equivalent to the $n \times n$ identity matrix I_n.

c. A has n pivot positions.

d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

e. The columns of A form a linearly independent set.

f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

g. The equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

h. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

j. There is an $n \times n$ matrix C such that $CA = I$

k. There is an $n \times n$ matrix D such that $AD = I$.

l. A^T is an invertible matrix.

⋆ If B is $n \times n$ and $AB = I$, then A and B are inverses of each other.

Exercise: Suppose H is a 5×5 matrix and suppose there is a vector $\mathbf{v} \in \mathbb{R}^5$ which is *not* a linear combination of the columns of H. What can you say about the number of solutions to $H\mathbf{x} = \mathbf{0}$? Explain.

Exercise: If C is 6×6 and the equation $C\mathbf{x} = \mathbf{v}$ has a solution for every $\mathbf{v} \in \mathbb{R}^6$, is it possible that for some $\mathbf{v}$, the equation $C\mathbf{x} = \mathbf{v}$ has more than one solution? Explain.

Exercise: Given an $n \times m$ matrix A, if $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, do the columns of A span $\mathbb{R}^n$? Explain.

Exercise: Suppose A is an $n \times n$ matrix with the property that the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Without using the IMT, explain directly why the equation $A\mathbf{x} = \mathbf{b}$ must have a solution for each $\mathbf{b} \in \mathbb{R}^n$.

Definition: A linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is **invertible** if there exists a function $S : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that

$$S(T(\mathbf{x})) = \mathbf{x} \quad \text{for all } \mathbf{x} \in \mathbb{R}^n$$

$$T(S(\mathbf{x})) = \mathbf{x} \quad \text{for all } \mathbf{x} \in \mathbb{R}^n$$

Theorem 9 Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear transformation and let A be the standard matrix for T. Then T is invertible if and only if A is an invertible matrix. In that case, the linear transformation S described above is the unique inverse of T, and $S(\mathbf{x}) = A^{-1}\mathbf{x}$.

Exercise: If A is an $n \times n$ matrix and the transformation $T(\mathbf{x}) = A\mathbf{x}$ is one-to-one, what else can you say about the transformation T? Explain.

Exercise: Suppose a linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ has the property that $T(\mathbf{u}) = T(\mathbf{v})$ for some pair of distinct (unequal) vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$. Can T map $\mathbb{R}^n$ onto $\mathbb{R}^n$? Explain.