Section 2.2 – The Inverse of a Matrix

Main Ideas in this section:

- Basic properties and terminology regarding A^{-1}
- Elementary row operations $\sim$ Elementary matrices
- Finding A^{-1}

Definitions:

1. An $n \times n$ matrix A is **invertible** (a.k.a **nonsingular**) if there is an $n \times n$ matrix C such that
$$CA = I_n = AC,$$
where I_n is the $n \times n$ identity matrix. In such a case, C is the *unique* inverse of A and is denoted by A^{-1}.

2. A matrix that is <u>not</u> invertible is said to be **singular**. All non-square matrices are singular by default.

Theorem 5 If an $n \times n$ matrix A is invertible, then for each $\mathbf{b}$ in $\mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the *unique* solution $\mathbf{x} = A^{-1}\mathbf{b}$.

Example: The matrix $A = \begin{bmatrix} 5 & 10 \\ 4 & 7 \end{bmatrix}$ is nonsingular, and $A^{-1} = \begin{bmatrix} -7/5 & 2 \\ 4/5 & -1 \end{bmatrix}$. Find the solution to $A\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Example: The augmented system $[\ A \mid \mathbf{b}\ ]$ given by $\begin{bmatrix} 1 & -2 & -1 & 0 \\ -1 & 5 & 6 & 1 \\ 5 & -4 & 5 & 0 \end{bmatrix}$ is row equivalent to $\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 3 & 5 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix}$. Is A an invertible matrix? Explain.

Useful properties of invertible matrices:

Theorem 6

a. If A is an invertible matrix, then A^{-1} is also invertible, and
$$(A^{-1})^{-1} = A$$

b. If A and B are $n \times n$ invertible matrices, then so is AB, and
$$(AB)^{-1} = B^{-1}A^{-1}$$

c. If A is an invertible matrix, then so is A^T, and
$$(A^T)^{-1} = (A^{-1})^T$$

Proof of b:

$$\text{From the left}: \; B^{-1}A^{-1} \cdot AB = B^{-1}I_nB = B^{-1}B = I_n.$$

$$\text{From the right}: \; AB \cdot B^{-1}A^{-1} = AI_nA^{-1} = AA^{-1} = I_n.$$

Definition: An **elementary matrix** is one that is obtained by performing a single elementary row operation on the identity matrix.

Example: Let elementary matrices E_1, E_2, and E_3 be given by $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -6 & 0 & 1 \end{bmatrix}$,
and let

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

a. Describe how each elementary matrix can be obtained from I_3 in a single elementary row operation.

b. Compute E_1A, E_2A, and E_3A and describe how these products can be obtained by elementary row operations on A.

Conclusion drawn from the last example:

If an elementary row operation is performed on an $m \times n$ matrix A, the resulting matrix can be written as EA, where the $m \times m$ matrix E is created by performing the same row operation on I_m.

$\star$ NOTE: Since elementary row operations are invertible, elementary matrices must be invertible too. In fact,...

The inverse of an elementary matrix E is the elementary matrix of the same type that transforms E back into I.

Example: Let $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -6 & 0 & 1 \end{bmatrix}$. Then $E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ +6 & 0 & 1 \end{bmatrix}$.

Finding Inverses – a simple algorithm based on this theorem:

Theorem 7 An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n. In this case, any sequence of elementary row operations that reduces A to I_n also transforms I_n to A^{-1}.

Proof:

Algorithm to find A^{-1}

Row reduce the augmented matrix $\begin{bmatrix} A & I \end{bmatrix}$. If A is row equivalent to I, then $\begin{bmatrix} A & I \end{bmatrix}$ is row equivalent to $\begin{bmatrix} I & A^{-1} \end{bmatrix}$. Otherwise, A is singular.

Example: Find the inverse of the matrix $A = \begin{bmatrix} -9 & 14 & -3 \\ -4 & 8 & -2 \\ 3 & -2 & 1 \end{bmatrix}$.

Solution:

$$\begin{aligned}
\begin{bmatrix} A & I \end{bmatrix} &= \begin{bmatrix} -9 & 14 & -3 & 1 & 0 & 0 \\ -4 & 8 & -2 & 0 & 1 & 0 \\ 3 & -2 & 1 & 0 & 0 & 1 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & -6 & 1 & 0 & -1 & -1 \\ -4 & 8 & -2 & 0 & 1 & 0 \\ -9 & 14 & -3 & 1 & 0 & 0 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & -6 & 1 & 0 & -1 & -1 \\ 0 & -16 & 2 & 0 & -3 & -4 \\ 0 & -40 & 6 & 1 & -9 & -9 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & -6 & 1 & 0 & -1 & -1 \\ 0 & 1 & -\frac{1}{8} & 0 & \frac{3}{16} & \frac{1}{4} \\ 0 & -40 & 6 & 1 & -9 & -9 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & 0 & \frac{1}{4} & 0 & \frac{1}{8} & \frac{1}{2} \\ 0 & 1 & -\frac{1}{8} & 0 & \frac{3}{16} & \frac{1}{4} \\ 0 & 0 & 1 & 1 & -\frac{3}{2} & 1 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{8} & 0 & \frac{3}{8} \\ 0 & 0 & 1 & 1 & -\frac{3}{2} & 1 \end{bmatrix}
\end{aligned}$$

So $A^{-1} = \begin{bmatrix} -1/4 & 1/2 & 1/4 \\ 1/8 & 0 & 3/8 \\ 1 & -3/2 & 1 \end{bmatrix}$.

Finding A^{-1} when A is 2×2:

Theorem 4 Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad - bc = 0$, then A is not invertible.

⋆*Note:* $ad - bc$ is called the *determinant* of A, and is denoted by $\det(A)$.

Example: Use Theorem 4 to solve the system

$$\begin{aligned} -7x_1 + 3x_2 &= 2 \\ 5x_1 - 2x_2 &= 1 \end{aligned}$$

Summary

- Formula/algorithm find A^{-1} for invertible/nonsingular A
- Elementary row operations $\sim$ Elementary matrices