Section 2.1 Matrix Operations

Key Ideas in this section

- Element-wise notation, basic definitions
- Properties of matrix-matrix multiplication, matrix transpose, matrix powers

Element-wise notation for an $m \times n$ matrix A is

$$\begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

In other words, a_{ij} is the entry in the ith row and jth column of A.

The **diagonal** entries in an $m \times n$ matrix $A = [a_{ij}]$ are $a_{11}, a_{22}, \ldots,$ and they form the **main diagonal** of A. A **diagonal matrix** is a square matrix whose off-diagonal elements are all zeros. (Example: I_n is a diagonal matrix.) The **zero matrix** is a matrix whose entries are all zero, and is denoted by 0. Two $m \times n$ matrices A and B are **equal** if $a_{ij} = b_{ij}$ for $i = 1 \cdots m$ and $j = 1 \cdots n$.

Simple Matrix Operations: $A + B$**,** $A - B$**,** cA

- If A and B are both size $m \times n$, then $A + B$ and $A - B$ are defined, and

$$A \pm B = \begin{bmatrix} a_{11} \pm b_{11} & \cdots & a_{1n} \pm b_{1n} \\ \vdots & & \vdots \\ a_{m1} \pm b_{m1} & \cdots & a_{mn} \pm b_{mn} \end{bmatrix}$$

- For any scalar c and any $m \times n$ matrix A, scalar multiplication cA is defined as

$$cA = \begin{bmatrix} c \cdot a_{11} & \cdots & c \cdot a_{1n} \\ \vdots & & \vdots \\ c \cdot a_{m1} & \cdots & c \cdot a_{mn} \end{bmatrix}$$

Theorem 1 Let A, B, and C be matrices of the same size, and let r and s be scalars.

a. $A + B = B + A$

b. $(A + B) + C = A + (B + C)$

c. $A + 0 = A$

d. $r(A + B) = rA + rB$

e. $(r + s)A = rA + sA$

f. $r(sA) = (rs)A$

Matrix Multiplication

A **composition** of mappings:

$$\mathbf{x} \quad \longrightarrow \quad B\mathbf{x} \quad \longrightarrow \quad A(B\mathbf{x})$$

Our goal: to represent the composition $x \mapsto A(B\mathbf{x})$ as a single matrix multiplication, $x \mapsto (AB)\mathbf{x}$.

Definition If A is an $m \times n$ matrix and B is an $n \times p$ matrix with columns $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_p$, then the product AB is the $m \times p$ matrix whose columns are $A\mathbf{b}_1, A\mathbf{b}_2, \ldots, A\mathbf{b}_p$. That is,

$$AB = A[\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_p] = [A\mathbf{b}_1 \quad A\mathbf{b}_2 \quad \cdots \quad A\mathbf{b}_p]$$

Example Let $A = \begin{bmatrix} 2 & 2 & 1 & 0 \\ -1 & 0 & 1 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ and $B = [\mathbf{b}_1 \ \mathbf{b}_2] = \begin{bmatrix} 2 & 2 \\ 1 & -1 \\ 4 & 0 \\ 0 & 1 \end{bmatrix}$. Compute $A\mathbf{b}_1$, $A\mathbf{b}_2$, and AB.

Row-Column Rule for computing AB.
If the product AB is defined, then the entry in row i and column j of AB is the sum of the products of corresponding entries from row i of A and column j of B. That is, if A is an $m \times n$ matrix and $(AB)_{ij}$ denotes the ij entry of AB, then

$$(AB)_{ij} = a_{i1}b1j + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

Example Compute $\begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & 1 \end{bmatrix}$ times $\begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 1 & 2 \end{bmatrix}$ using the Row-Column rule for matrix multiplication.

Theorem 2 – Properties of Matrix Multiplication
Let A be an $m \times n$ matrix, and let B and C have sizes for which the indicated sums and products are defined. Then

a. $A(BC) = (AB)C$ (Associative Law of Multiplication)
b. $A(B + C) = AB + AC$ (Left Distributive Law)
c. $(B + C)A = BA + CA$ (Right Distributive Law)
d. $r(AB) = (rA)B = A(rB)$
for any scalar r
e. $I_m A = A = AI_n$ (Identity for Matrix Multiplication)

Beware!! Watch out for some pitfalls:

1. There is NO commutative law for matrix multiplication. That is, generally $AB \neq BA$, and we say that matrices don't "commute."

2. Regular algebraic-style cancellation laws do NOT hold for matrix multiplication. That is, if $AB = AC$, then it is not true in general that $B = C$.

3. If $AB = 0$, it is NOT necessarily true that A or B must be 0.

Other Definitions

- **Powers** of a matrix: If A is a square matrix and if k is a positive integer, then A^k denotes the product of k copies of A:
$$A^k = \underbrace{A \cdots A}_{k}.$$
A^0 is interpreted as the identity matrix.

- **Transpose** of a matrix: Given an $m \times n$ matrix A, the transpose of A is the $n \times m$ matrix, denoted by A^T, whose columns are formed from the corresponding rows of A.

Theorem 3 Let A and A_i denote matrices whose sizes are appropriate for the following sums and products. Then

a. $(A^T)^T = A$

b. $(A_1 + A_2 + \cdots + A_p)^T = A_1^T + A_2^T + \cdots + A_p^T$

c. For any scalar r, $(rA)^T = rA^T$

d. $(A_1 A_2 \cdots A_p)^T = A_p^T \cdots A_2^T A_1^T$