Section 1.9 The Matrix of a Linear Transformation

Key Concepts

- Every linear transformation $T:\mathbb{R}^n \to \mathbb{R}^m$ is actually a matrix transformation $\mathbf{x} \mapsto A\mathbf{x}$.
- The action of T on unit vectors $\mathbf{e}_i, i = 1 \dots m$, determines the structure of A.
- Important properties of T (one-to-one, onto) are intimately related to known properties of A.

Example Given unit vectors $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $\dots$, $\mathbf{e}_m = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$,

any $\mathbf{x} \in \mathbb{R}^m$ can be written as

$$\begin{aligned} \mathbf{x} &= x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_m\mathbf{e}_m \\ &= \underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}}_{I_m} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}}_{\mathbf{x}} \\ &= I_m\mathbf{x}. \end{aligned}$$

Recall from section 1.8: if $T:\mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then

$$T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v}).$$

General result:

$$T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + \cdots + c_pT(\mathbf{v}_p).$$

Example Suppose T is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^4$ with

$$T(\mathbf{e}_1) = \begin{bmatrix} 2 \\ -3 \\ 4 \\ 5 \end{bmatrix}, \quad T(\mathbf{e}_2) = \begin{bmatrix} 5 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \quad \text{and} \quad T(\mathbf{e}_3) = \begin{bmatrix} -1 \\ -2 \\ 0 \\ 7 \end{bmatrix}.$$

Compute $T(\mathbf{x})$ for any $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$ and determine the matrix A that implements the transformation T.

Question: What is the size of the matrix A that corresponds to the map $T : \mathbb{R}^p \rightarrow \mathbb{R}^q$?

Theorem 10 Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation. Then there exists a unique $m \times n$ matrix A such that

$$T(\mathbf{x}) = A\mathbf{x} \text{ for all } \mathbf{x} \text{ in } \mathbb{R}^n.$$

In fact, A is the $m \times n$ matrix whose jth column is the vector $T(\mathbf{e}_j)$, with $\mathbf{e}_j \in \mathbb{R}^n$:

$$A = [T(\mathbf{e}_1)\ T(\mathbf{e}_2)\ \cdots T(\mathbf{e}_n)]$$

The matrix A is called the **standard matrix for the linear transformation** T.

Example Determine the standard matrices for the following linear transformations $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$.

Reflection across x_1 axis

Reflection across x_2 axis

Reflection across line $x_2 = x_1$

Reflection across line $x_2 = -x_1$

Example Find the standard matrix for $T : \mathbb{R}^2 \to \mathbb{R}^3$ if $T : \mathbf{x} \mapsto \begin{bmatrix} x_1 - 2x_2 \\ 4x_1 \\ 3x_1 + 2x_2 \end{bmatrix}$.

Example Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that rotates each point in $\mathbb{R}^2$ about the origin through and angle $\pi/4$ radians (counterclockwise). Determine the standard matrix for T.

Question: Determine the standard matrix for the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that rotates each point in $\mathbb{R}^2$ counterclockwise around the origin through an angle of ϕ radians.

Definitions:

- A mapping $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is **onto** $\mathbb{R}^m$ if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of *at least one* $\mathbf{x}$ in $\mathbb{R}^n$.

  Equivalent ways to state this definition are:

  Note: T is *not* onto $\mathbb{R}^m$ when there is some $\mathbf{b}$ in $\mathbb{R}^m$ for which the equation $T(\mathbf{x}) = \mathbf{b}$ has no solution, or equivalently, whenever the is some $\mathbf{b}$ in $\mathbb{R}^m$ for which the system $[A|\mathbf{b}]$ is inconsistent.

- A mapping $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is **one to one** if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of *at most one* $\mathbf{x}$ in $\mathbb{R}^n$.

Example Let T be the transformation whose standard matrix is

$$A = \begin{bmatrix} 1 & 3 & -4 & 1 \\ 0 & 2 & -3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

Does T map $\mathbb{R}^4$ onto $\mathbb{R}^3$? Is T a one-to-one mapping? Explain your reasoning.

Theorem 11 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is one-to-one if and only if the equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

Proof:

Theorem 12 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let A be the standard matrix for T. Then:

a. T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$.

b. T is one-to-one if and only if the columns of A are linearly independent.

   Proof:

Summary of section 1.9

- Each $T : \mathbb{R}^n \to \mathbb{R}^m$ corresponds to standard matrix A of size $m \times n$.
- The standard matrix is determined by the action of T on the standard unit vectors $\mathbf{e}_i$.
- Properties of T (onto, one-to-one) are deeply related to questions of existence/uniqueness of solutions to $A\mathbf{x} = \mathbf{b}$.