Section 1.9 The Matrix of a Linear Transformation

Key Concepts

- Every linear transformation \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is actually a matrix transformation \( x \mapsto Ax \).
- The action of \( T \) on unit vectors \( e_i, i = 1 \ldots m \), determines the structure of \( A \).
- Important properties of \( T \) (one-to-one, onto) are intimately related to known properties of \( A \).

**Example** Given unit vectors 
\[
\begin{align*}
\mathbf{e}_1 &= \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \end{bmatrix}, \\
\mathbf{e}_2 &= \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \end{bmatrix}, \\
\vdots \\
\mathbf{e}_m &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix},
\end{align*}
\]

any \( \mathbf{x} \in \mathbb{R}^m \) can be written as
\[
\mathbf{x} = x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2 + \cdots + x_m \mathbf{e}_m = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = I_m \mathbf{x}.
\]

Recall from section 1.8: if \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is a linear transformation, then
\[
T(cu + dv) = cT(u) + dT(v).
\]

General result:
\[
T(c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \cdots + c_p \mathbf{v}_p) = c_1 T(\mathbf{v}_1) + c_2 T(\mathbf{v}_2) + \cdots + c_p T(\mathbf{v}_p).
\]

**Example** Suppose \( T \) is a linear transformation from \( \mathbb{R}^3 \) to \( \mathbb{R}^4 \) with
\[
\begin{align*}
T(\mathbf{e}_1) &= \begin{bmatrix} 2 \\ -3 \\ 4 \\ 5 \end{bmatrix}, \\
T(\mathbf{e}_2) &= \begin{bmatrix} 5 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \text{ and } T(\mathbf{e}_3) &= \begin{bmatrix} -1 \\ -2 \\ 0 \\ 7 \end{bmatrix}.
\end{align*}
\]

Compute \( T(\mathbf{x}) \) for any \( \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \) and determine the matrix \( A \) that implements the transformation \( T \).

\[
A = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & T(\mathbf{e}_3) \end{bmatrix} = \begin{bmatrix} 2 & 5 & -1 \\ -3 & 0 & -2 \\ 4 & 1 & 0 \\ 5 & -1 & 0 \end{bmatrix}
\]

So \( \mathbf{Ax} = \mathbf{Ax} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 + 5x_2 - x_3 \\ -3x_1 + 0x_2 - 2x_3 \\ 4x_1 + x_2 + 0x_3 \\ 5x_1 - x_2 + 7x_3 \end{bmatrix} \).
Question: What is the size of the matrix $A$ that corresponds to the map $T: \mathbb{R}^p \to \mathbb{R}^q$?

$$T: \mathbb{R}^p \to \mathbb{R}^q$$

so $A$ is a $q \times p$ matrix

**Theorem 10** Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique $m \times n$ matrix $A$ such that

$$T(x) = Ax$$

for all $x$ in $\mathbb{R}^n$.

In fact, $A$ is the $m \times n$ matrix whose $j$th column is the vector $T(e_j)$, with $e_j \in \mathbb{R}^n$:

$$A = [T(e_1) \ T(e_2) \cdots T(e_n)]$$

The matrix $A$ is called the **standard matrix for the linear transformation** $T$.

**Example** Determine the standard matrices for the following linear transformations $T: \mathbb{R}^2 \to \mathbb{R}^2$.

Reflection across $x_1$ axis

So $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ is the standard matrix for $T$.

Reflection across $x_2$ axis

So $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ is the standard matrix for $T$.

Reflection across line $x_2 = x_1$

So $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is the standard matrix for $T$.

Reflection across line $x_2 = -x_1$

So $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is the standard matrix for $T$. 


Example Find the standard matrix for $T : \mathbb{R}^2 \to \mathbb{R}^3$ if $T : \mathbf{x} \mapsto \begin{bmatrix} x_1 - 2x_2 \\ 4x_1 \\ 3x_1 + 2x_2 \end{bmatrix}$.

$$A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 2x_2 + 0x_3 \\ 4x_1 + 0x_1 + 0x_3 \\ 3x_1 + 2x_2 + 0x_3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 0 \\ 4 & 0 & 0 \\ 3 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

So $A = \begin{bmatrix} 1 & -2 & 0 \\ 4 & 0 & 0 \\ 3 & 2 & 0 \end{bmatrix}$ is the standard matrix for $T$.

Example Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that rotates each point in $\mathbb{R}^2$ about the origin through and angle $\pi/4$ radians (counterclockwise). Determine the standard matrix for $T$.

\[ T(e_1) = \left( \cos(\pi/4), \sin(\pi/4) \right) = \left( \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right) \]

\[ T(e_2) = \left( \cos(3\pi/4), \sin(3\pi/4) \right) = \left( -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right) \]

So, $A = \begin{bmatrix} T(e_1) \\ T(e_2) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$ is the standard matrix for $T$.

Question: Determine the standard matrix for the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that rotates each point in $\mathbb{R}^2$ counterclockwise around the origin through an angle of $\phi$ radians.

\[ T(e_1) = \left( \cos(\phi), \sin(\phi) \right) \]

\[ T(e_2) = \left( \cos(\phi), \sin(\phi) \right) \]

So $A = \begin{bmatrix} T(e_1) \\ T(e_2) \end{bmatrix} = \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix}$ is the standard matrix for $T$. 

3
Definitions:

- A mapping \( T : \mathbb{R}^n \to \mathbb{R}^m \) is **onto** \( \mathbb{R}^m \) if each \( b \) in \( \mathbb{R}^m \) is the image of at least one \( x \) in \( \mathbb{R}^n \).

  Equivalent ways to state this definition are:

  For each \( b \in \mathbb{R}^m \), there is at least one \( x \in \mathbb{R}^n \) such that \( T(x) = b \).

Note: \( T \) is *not* onto \( \mathbb{R}^m \) when there is some \( b \) in \( \mathbb{R}^m \) for which the equation \( T(x) = b \) has no solution, or equivalently, whenever the is some \( b \) in \( \mathbb{R}^m \) for which the system \([A|b]\) is inconsistent.

- A mapping \( T : \mathbb{R}^n \to \mathbb{R}^m \) is **one to one** if each \( b \) in \( \mathbb{R}^m \) is the image of at most one \( x \) in \( \mathbb{R}^n \).

  For each \( b \in \mathbb{R}^m \), there is at most one \( x \in \mathbb{R}^n \) such that \( T(x) = b \).

**Example** Let \( T \) be the transformation whose standard matrix is

\[
A = \begin{bmatrix}
1 & 3 & -4 & 1 \\
0 & 2 & -3 & 0 \\
0 & 0 & 0 & 3
\end{bmatrix}.
\]

Does \( T \) map \( \mathbb{R}^4 \) onto \( \mathbb{R}^3 \)? Is \( T \) a one-to-one mapping? Explain your reasoning.

\( T \) is onto \( \mathbb{R}^2 \) \iff \( \text{For each } b \in \mathbb{R}^3, \text{ there is at least one } x \in \mathbb{R}^4 \text{ such that } T(x) = b \).

\( \iff \) \( \text{For each } b \in \mathbb{R}^3, \text{ } T(x) = b \text{ has at least one solution} \).

\( \iff \) Every \( b \in \mathbb{R}^3 \) is in the span of the columns of \( A \).

\( \iff \) The columns of \( A \) span \( \mathbb{R}^2 \) (as, use Theorem 12.2 below to get to this point).

\( A \) spans \( \mathbb{R}^2 \) \iff \( A \) has a pivot position in each row.

\( A \) does have a pivot position in each row (seen from echelon form), so yes, \( A \) is onto.
$T$ is one-to-one $\iff$ for every $b \in \mathbb{R}^2$, $Ax=b$ has at most one solution.

$A$ has a free variable, $k$, so $Ax=0$ has infinitely many solutions (which is more than one).

So, $A$ is not one-to-one.
Theorem 11 Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation. Then $T$ is one-to-one if and only if the equation $T(x) = 0$ has only the trivial solution.

Proof:

$\Rightarrow$ Suppose $T$ is one-to-one. Since $T$ is linear, $T(0) = 0$. Since $T$ is one-to-one, $Tx = 0$ cannot have any other solutions. So the only solution to $Tx = 0$ is the trivial solution.

$\Leftarrow$ Consider the contrapositive. Suppose $T$ is not one-to-one. Then there are some $u, v \in \mathbb{R}^n$, $u \neq v$, such that $T(u) = T(v)$. Then $T(u-v) = T(u) - T(v)$, because $T$ is linear.

So, $T(u-v) = 0$, but $u-v \neq 0$ since $u \neq v$. So $Tx = 0$ has a nontrivial solution.

Theorem 12 Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation and let $A$ be the standard matrix for $T$. Then:

a. $T$ maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of $A$ span $\mathbb{R}^m$.

b. $T$ is one-to-one if and only if the columns of $A$ are linearly independent.

Proof:

a. $T$ is onto $\mathbb{R}^m$.

\[ \iff \text{For each } b \in \mathbb{R}^m, \text{ there is at least one } x \in \mathbb{R}^n \text{ such that } T(x) = b. \]

\[ \iff \text{For each } b \in \mathbb{R}^m, \text{ } T(x) = b \text{ has at least one solution.} \]

\[ \iff \text{Every } b \in \mathbb{R}^m \text{ is in the span of the columns of } A \]

\[ \iff \text{The columns of } A \text{ span } \mathbb{R}^m. \]

b. $T$ is one-to-one $\iff Tx = 0$ has only the trivial solution (by above).

\[ \iff \text{The columns of } A \text{ are linearly independent (8.1.7).} \]
Summary of section 1.9

- Each \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) corresponds to standard matrix \( A \) of size \( m \times n \).
- The standard matrix is determined by the action of \( T \) on the standard unit vectors \( e_i \).
- Properties of \( T \) (onto, one-to-one) are deeply related to questions of existence/uniqueness of solutions to \( Ax = b \).