Spanning and Linear Independence

Definition 1 *Let* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ *and* $\mathbf{b}$ *be vectors in* $\mathbb{R}^m$. *Then* a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ *is any sum*

$$\mathbf{b} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + \cdots + c_p\mathbf{v}_p \quad \text{where} \quad c_1, c_2, \ldots, c_p \text{ are real constants}$$

The span of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$, *written* $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$, *is the set of* **all** *linear combinations of the vectors* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$.

1. To determine whether or not $\mathbf{b}$ is in $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ you must find out if there are scalars $c_1, c_2, \ldots, c_p$ so that $\mathbf{b} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$. This vector equation is equivalent to the matrix equation with augmented matrix $[\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \cdots\ \mathbf{v}_p\ \mathbf{b}]$[1]

   To check this, set up the augmented matrix $[\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \cdots\ \mathbf{v}_p\ \mathbf{b}]$ and reduce it to row echelon form.

   - If there is **no** row of the form $[0\ 0\ 0\ \cdots 0\ \ \text{non-zero number}]$ then the system is consistent and there are solutions for the c's. Therefore, $\mathbf{b}$ *is in* $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$. If you are asked to find scalars that make this true, you must continue to reduced row echelon form to find them.

   - If there **is** a row of the form $[0\ 0\ 0\ \cdots\ 0\ \ \text{non-zero number}]$ then the system is inconsistent and $\mathbf{b}$ is *not* in $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$.

2. To determine whether or not $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ spans $\mathbb{R}^m$ you must find out if there are scalars $c_1, c_2, \ldots, c_p$ so that $\mathbf{y} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$ for **any** vector $\mathbf{y}$ in $\mathbb{R}^m$.

   To check this, set up the matrix $A = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \cdots\ \mathbf{v}_p]$ and reduce it to row echelon form. If there is a pivot position in every row, i.e. the row echelon form of A has no zero rows (so $A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b}$), then the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ does span $\mathbb{R}^m$. If not, then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ does not span $\mathbb{R}^m$

An *indexed set of vectors* is a set of vectors, $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^m$ for which order matters. As indexed sets $\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix}\right\}$ and $\left\{\begin{bmatrix}3\\4\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}\right\}$ are different, although they are the same as sets that are *not indexed*.

Definition 2 *, An indexed set of vectors* $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ *in* $\mathbb{R}^m$ *is said to be* **linearly independent** *if the vector equation*

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution, $x_1 = x_2 = \cdots = x_p = 0$.

Equivalently, if A is an $m \times n$ matrix, then the columns of the matrix A are linearly independent if the matrix equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$ where $\mathbf{0}$ is the zero vector in $\mathbb{R}^m$.
On the other hand, if there are specific scalars $c_1, c_2, \ldots c_p$ not all zero such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$$

then the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ *is said to be* **linearly dependent**.
If $c_1, c_2, \ldots c_p$ are scalars that are not all equal to zero and $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$ holds, then $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$ is called a **dependence relation**.

Please Turn Over

[1]Often an augmented matrix is written ($[A \mid \mathbf{b}]$, but our book does not use a $|$.
(These are slightly edited versions of notes from Mary Glaser.)

To determine whether or not $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is linearly independent, you determine whether $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$ has only the trivial solution or nontrivial solutions. Equivalently, you can ask the same question for $A\mathbf{x} = \mathbf{0}$ has where $A = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p]$.

- If the only solution to $A\mathbf{x} = \mathbf{0}$ is the trivial solution, then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is linearly independent.
- If there are nontrivial solutions to $A\mathbf{x} = \mathbf{0}$ then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is linearly dependent.

Recall that, to solve $A\mathbf{x} = \mathbf{0}$, one often row reduces A (instead of $[A\ \mathbf{0}]$) and converts to a homogenous system. We will use this convention in this description.

To check independence, set up the matrix $A = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \cdots\ \mathbf{v}_p]$ and reduce it to row echelon form.

- If every column is a pivot column, then the vector equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$ has no free variables, so the only solution to is the trivial one is $x_1 = x_2 = \cdots = x_p = 0$ (the only solution to the matrix equation $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$.)
- If the vector equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$ (or matrix equation $A\mathbf{x} = \mathbf{0}$) has non-trivial solutions, then the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is linearly dependent.
  To find dependence relations, you must transform the matrix $[\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \cdots\ \mathbf{v}_p]$ to reduced row echelon form. In this case there will be an infinite number of solutions because the equations $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$ will have free variables. By choosing specific values for the x's, you can find specific solutions. Each nontrivial solution represents a dependence relation among the vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$.