Spanning and Linear Independence

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ and $\mathbf{b}$ be vectors in $\mathbb{R}^m$. Let A be the m $\times$ p matrix whose columns are $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$.

<u>dfn:</u> Span$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ = the set of **all** linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$.

1. To determine whether or not $\mathbf{b}$ is in Span$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ you must find out if there are scalars $c_1, c_2, \ldots, c_p$ so that $\mathbf{b} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$.

   Mechanically, set up the augmented matrix $[\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \cdots\ \mathbf{v}_p \,|\, \mathbf{b}]$ and reduce it to row echelon form. If there is **no** row of the form $[0\ 0\ 0\ \cdots\ 0 \,|$ non-zero number$]$ then $\mathbf{b}$ is in Span$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$. If you are asked to find scalars that make this true, you must continue to reduced row echelon form to find them.

2. To determine whether or not $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ span $\mathbb{R}^m$ you must find out if there are scalars $c_1, c_2, \ldots, c_p$ so that $\mathbf{y} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$ for **any** vector $\mathbf{y}$ in $\mathbb{R}^m$.

   Mechanically, set up the matrix $[\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \cdots\ \mathbf{v}_p]$ and reduce it to row echelon form. If there is a pivot position in every row, i.e. the row echelon form of A has no zero rows, then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ do span $\mathbb{R}^m$.

<u>dfn:</u> An indexed set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^m$ is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution, $x_1 = x_2 = \cdots = x_p = 0$. Equivalently, the columns of the matrix A are linearly independent if the matrix equation $A\mathbf{x} = \mathbf{0}$ has the trivial solution $\mathbf{x} = \mathbf{0}$ where $\mathbf{0}$ is the zero vector in $\mathbb{R}^p$.
On the other hand, if there are specific scalars $c_1, c_2, \ldots c_p$ not all zero such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$$

then the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is said to be **linearly dependent** and the equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$ where $c_1, c_2, \ldots c_p$ are specific scalars is called a **dependence relation**.

To determine whether or not $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is linearly independent, you must find out what kind of solutions $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$ (or equivalently $A\mathbf{x} = \mathbf{0}$) has.

Mechanically, set up the augmented matrix $[\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \cdots\ \mathbf{v}_p \,|\, \mathbf{0}]$ and reduce it to row echelon form.
If every column is a pivot column, meaning the vector equation (matrix equation) has no free variables, then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is linearly independent because the only solution to the vector equation is
$x_1 = x_2 = \cdots = x_p = 0$ (the only solution to the matrix equation is $\mathbf{x} = \mathbf{0}$.)
Note: Since the rightmost column of $[\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \cdots\ \mathbf{v}_p \,|\, \mathbf{0}]$ always contains zeroes throughout the row reduction process, it is often omitted.

If you determine that the vector equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$ (or matrix equation $A\mathbf{x} = \mathbf{0}$) has nontrivial solutions, then you must transform $[\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \cdots\ \mathbf{v}_p \,|\, \mathbf{0}]$ to reduced row echelon form to find them. In this case there will be an infinite number of solutions because the equations will have free variables. By choosing specific parameters you can find specific solutions. Each solution represents a dependence relation among the vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$.