§4.5 The Dimension of a Vector Space

In this section we classify vector spaces by the sizes of their bases.

Theorem 9 If a vector space V has a basis $\mathcal{B}$ of n vectors, $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$, then any set in V containing more than n vectors must be linearly dependent.

NB: An immediate consequence of Theorem 9 is that if $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ is a basis for V, then any linearly independent set in V must contain at most n vectors.

Theorem 10 If a vector space V has a basis of n vectors, then every basis of V must contain exactly n vectors.

dfn: If the vector space V is spanned by a finite set of vectors, then V is said to be **finite-dimensional**. If V is finite-dimensional but not the zero vector space, then the **dimension** of V, written dim V, is the number of vectors in a basis for V.[1] The dimension of the zero vector space, $V = \{\mathbf{0}\}$, is defined to be 0. If V is not spanned by any finite set, then V is said to be **infinite-dimensional**.

Vector Space	Basis	Dimension
$\mathbb{R}^n$	$\mathcal{E} = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$	n
$\mathbb{P}_n$	$\mathcal{B} = \{1, t, t^2, \ldots, t^n\}$	$n + 1$
$M_{2\times 2}$	$\mathcal{B} = \{\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)\}$	4
$\mathbb{P}$	$\mathcal{B} = \{1, t, t^2, t^3, \ldots\}$	infinite
V = real sequences	$\mathcal{B} = \{(1,0,0,\ldots), (0,1,0,\ldots), (0,0,1,\ldots), \ldots\}$	infinite

Subspaces of a Finite-Dimensional Vector Space

Theorem 11 Let H be a subspace of a finite-dimensional vector space V. Any linearly independent set in H can be expanded, if necessary, to a basis for H. Also, H is finite-dimensional and

$$\dim H \leq \dim V$$

Theorem 12 Let V be a p-dimensional vector space, $p \geq 1$. Any linearly independent set of exactly p vectors in V is automatically a basis for V. Any set of exactly p vectors that span V is automatically a basis for V.

This theorem saves us work! If we know the dimension of a finite-dimensional vector space V, then to show that a set of the right size is a basis for V we need only show that it is linearly independent *or* that it spans V. In many cases, showing linear independence is easier than showing spanning.

The Dimensions of Nul A and Col A: Suppose A is an $n \times m$ matrix. To find a spanning set for Nul A we solve $A\mathbf{x} = \mathbf{0}$. If this equation has k free variables then our spanning set contains k vectors, $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$. But these vectors are also linearly independent and hence form a basis for Nul A. The pivot columns of A form a basis for Col A.

$$\text{dim Nul } A = \text{the number of free variables in the equation } A\mathbf{x} = \mathbf{0}.$$

$$\text{dim Col } A = \text{the number of pivot columns of } A.$$

[1]Theorem 10 shows that every basis of V has the same number of vectors, so $\dim V$ is unambiguously defined.