## §4.4 Coordinate Systems

In general, people are more comfortable working with $\mathbb{R}^n$ and its subspaces than with other types of vector spaces and subspaces. The goal in this section is to define an amazing correspondence between $\mathbb{R}^n$ and other vector spaces that allows us to do calculations in $\mathbb{R}^n$ to learn about more general vector spaces.

**Theorem 7 The Unique Representation Theorem**

Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ be a basis for a vector space $V$. Then for each vector $\mathbf{x} \in$ V, there exists a *unique* set of scalars $c_1, \ldots, c_n$ such that $\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$.

**dfn**: Suppose $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ is a basis for a vector space $V$ and $\mathbf{x} \in V$. The **coordinates of x relative to the basis** $\mathcal{B}$, or the $\mathcal{B}$**-coordinates of x**, are the weights $c_1, \ldots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n.$$

In this case, the vector $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ is called the **coordinate vector of x relative to** $\mathcal{B}$, or the $\mathcal{B}$**-coordinate vector of x**. Notice that $[\mathbf{x}]_{\mathcal{B}}$ is a vector in $\mathbb{R}^n$.

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### Coordinates in $\mathbb{R}^n$

When we are working in $\mathbb{R}^n$ and we write $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, we have written the coordinates of our vector $\mathbf{x}$ with respect to the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ of $\mathbb{R}^n$. We will now discuss how to compute a change of coordinates from an arbitrary basis for $\mathbb{R}^n$ to the standard basis and visa-versa.

Given a basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ for $\mathbb{R}^n$, define the matrix $P_{\mathcal{B}} = [\ \mathbf{b}_1\ \ \mathbf{b}_2\ \ \cdots\ \ \mathbf{b}_n\ ]$. Then if $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$, the vector equation $\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$ is equivalent to the matrix equation

$$\mathbf{x} = P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}.$$

This formula allows us to find $\mathbf{x}$ from its coordinates in the $\mathcal{B}$ basis. We call $P_{\mathcal{B}}$ the **change of coordinates matrix** from $\mathcal{B}$ to the standard basis in $\mathbb{R}^n$.

Since $P_{\mathcal{B}}$ is a square matrix with linearly independent columns, $P_{\mathcal{B}}^{-1}$ exists by the Invertible Matrix Theorem, so we can conclude that

$$[\mathbf{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}\mathbf{x}.$$

Therefore $P_{\mathcal{B}}^{-1}$ is the change of coordinates matrix from the standard basis in $\mathbb{R}^n$ to the basis $\mathcal{B}$.

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### Isomorphisms and Coordinate Mappings

**dfn**: Let $T : V \to W$ be a linear transformation. Then $T$ is an *isomorphism* if $T$ is one-to-one and $T$ is onto.
(Recall $T$ *is one-to-one:* if, for all $\mathbf{u}$ and $\mathbf{v}$ in $V$, if $T(\mathbf{u}) = T(\mathbf{v})$, then $\mathbf{u} = \mathbf{v}$ (equivalently, since $T$ is linear, $\mathbf{x} = \mathbf{0}$ is the only solution to $T(\mathbf{x}) = \mathbf{0}$). $T$ *is onto:* if for every $\mathbf{w} \in W$ there is a $\mathbf{v} \in V$ with $T(\mathbf{v}) = \mathbf{w}$.)

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Choosing an ordered basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ for a vector space $V$ introduces a coordinate system in $V$. That is, the coordinate mapping

$$\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}} \qquad \text{(sometimes written } C(\mathbf{x}) = [\mathbf{x}]_{\mathcal{B}}\text{)}$$

connects the possibly unfamiliar space $V$ with the familiar space $\mathbb{R}^n$. Points in $V$ can now be identified by their coordinates in $\mathbb{R}^n$, and every vector-space calculation in $V$ is accurately reproduced in $\mathbb{R}^n$ (and vice versa–see the Coordinate Map Theorem on the next page!). Note that $V$ is *not* $\mathbb{R}^n$ but $V$ does look like $\mathbb{R}^n$ as a vector space.

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**Theorem 8**
Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ be a basis for a vector space $V$. Then the coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is an isomorphism from $V$ to $\mathbb{R}^n$ (i.e., linear, one-to-one, and onto).

Remarks:

- Linearity of the coordinate mapping means for all choices of weights and vectors in $V$, that

$$[c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p]_{\mathcal{B}} = c_1[\mathbf{u}_1]_{\mathcal{B}} + \cdots + c_p[\mathbf{u}_p]_{\mathcal{B}}.$$

- Two isomorphic spaces (that is, two spaces linked by a one-to-one, onto linear transformation) may look like entirely different spaces, but as vector spaces they act the same!! This is evidenced by the fact that every vector space calculation in one space is accurately reproduced in the other, as in the parallel universes of $\mathbb{P}_2$ and $\mathbb{R}^3$:

**Example:** *The parallel universes of* $\mathbb{P}_2$ *and* $\mathbb{R}^3$*:* $\mathbb{P}_2$ is isomorphic to $\mathbb{R}^3$ by the coordinate map $\mathbf{p} \mapsto [\mathbf{p}]_{\mathcal{E}}$ where $\mathcal{E} = \{1, t, t^2\}$ is the standard basis in $\mathbb{P}_2$:

| $\mathbb{P}_2$ | $\mathbb{R}^3$ |
|---|---|
| $\mathbf{p}(t) = a + bt + ct^2$ | $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ |
| $(-1 + 2t - 3t^2) + (2 + 3t + 5t^2)$ $= 1 + 5t + 2t^2$ | $\begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}$ |
| $4(5t - 6t^2) = 20t - 24t^2$ | $4\begin{bmatrix} 0 \\ 5 \\ -6 \end{bmatrix} = \begin{bmatrix} 0 \\ 20 \\ -24 \end{bmatrix}$ |

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**Coordinate Map Theorem** Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ be a basis for a vector space $V$. Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be an indexed set of vectors in $V$. Then,

i. $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is independent in $V$ if and only if the set of coordinate vectors $\{[\mathbf{v}_1]_{\mathcal{B}}, \ldots, [\mathbf{v}_p]_{\mathcal{B}}\}$ is independent in $\mathbb{R}^n$.

ii. $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ spans $V$ if and only if the set of coordinate vectors $\{[\mathbf{v}_1]_{\mathcal{B}}, \ldots, [\mathbf{v}_p]_{\mathcal{B}}\}$ spans $\mathbb{R}^n$.

iii. the vector $\mathbf{w} \in V$ is a linear combination of $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ if and only if the coordinate vector $[\mathbf{w}]_{\mathcal{B}}$ is a linear combination of $\{[\mathbf{v}_1]_{\mathcal{B}}, \ldots, [\mathbf{v}_p]_{\mathcal{B}}\}$.

**Moral:** If $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ is a basis for $V$ and $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ an indexed set of vectors in $V$, then

- To check if $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is independent in $V$, check if $\{[\mathbf{v}_1]_{\mathcal{B}}, \ldots, [\mathbf{v}_p]_{\mathcal{B}}\}$ is independent in $\mathbb{R}^n$.
- To check if $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ spans $V$, check if $\{[\mathbf{v}_1]_{\mathcal{B}}, \ldots, [\mathbf{v}_p]_{\mathcal{B}}\}$ span $\mathbb{R}^n$.
- To check if $\mathbf{w} \in V$ is a linear combination of $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$, check if $[\mathbf{w}]_{\mathcal{B}}$ is a linear combination of $\{[\mathbf{v}_1]_{\mathcal{B}}, \ldots, [\mathbf{v}_p]_{\mathcal{B}}\}$.