§4.3 Linearly Independent Sets; Bases

<u>**dfn**</u>: A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_p\}$ in a vector space V is said to be **linearly independent** if the vector equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$ has only the trivial solution $x_1 = x_2 = \cdots = x_p = 0$.

The set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_p\}$ is called **linearly dependent** if there are scalars $c_1, c_2, \ldots, c_p$ not all equal to 0 such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$

- Any set containing the zero vector is linearly dependent.
- A set of two vectors is linearly dependent if and only if one is a multiple of the other.

Theorem 4 An indexed set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ of two or more vectors with $\mathbf{v}_1 \neq \mathbf{0}$ is linearly dependent if and only if some vector $\mathbf{v}_j$, $(j > 1)$ is a linear combination of the preceding vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{j-1}$.

<u>**dfn**</u>: Let H be a subspace of a vector space V. (Recall that $H = V$ is a possibility). An indexed set of vectors $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_p\}$ is called a **basis** for H if

1. $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_p\}$ is linearly independent.
2. $\mathcal{B}$ spans H. In other words, $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_p\}$.

Standard Bases:

- The *standard basis* for the real vector space $\mathbb{R}^n$ is $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ where $\mathbf{e}_i$ is the i^{th} column of I_n.
- The *standard basis* for the polynomial space $\mathbb{P}_n$ is $\mathcal{E} = \{1, t, t^2, \ldots, t^n\}$. (NB: there are $n+1$ vectors in the basis.)

Theorem 5 The Spanning Set Theorem

Let V be a vector space and let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ be a set in V and let $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$.

- If one of the vectors in S, for example $\mathbf{v}_k$, is a linear combination of the remaining vectors in S, then the set formed by removing $\mathbf{v}_k$ from S still spans H.
- If $H \neq \mathbf{0}$, then some subset of S is a basis for H.

NB: The spanning set theorem leads directly to a common method for finding a basis. Start with a spanning set S and "cast out" unwanted vectors: toss out $\mathbf{0}$ if it's in S and then (one by one) toss out any vector that can be written as a linear combination of the remaining vectors in S; keep tossing vectors that are linear combinations of the others until no vector is a linear combination of the others–then you have a basis. A basis for a vector space is a spanning set that is as small as possible. In other words, *a basis is a minimal spanning set.*

To build a basis from an independent set, test to see if the set spans the vector space. If not, keep adding vectors that keep the set linearly independent until it spans. A basis is a linearly independent set that is as large as possible. In other words, *a basis is a maximal linearly independent set.*

How to find bases for $\text{Nul}\, A$ **and** $\text{Col}\, A$ **where** $A = [\mathbf{a}_1 \cdots \mathbf{a}_n]$ **is an** $m \times n$ **matrix:**

- $\text{Nul}\, A$: Solve the homogeneous equation $A\mathbf{x} = \mathbf{0}$ and write the solution in parametric vector form. If the solution is not just $\mathbf{0}$, then write down the solution vectors corresponding to the free variables of the equation (each free variable gives a solution vector by setting it equal to 1 and the rest = 0). We have seen these vectors are automatically linearly independent. Hence these vectors form a basis for Nul A.
  (NB: They are not unique; any one of them can be replaced by a scalar multiple of itself. In fact this is often done to "beautify" the basis, e.g. replace a basis vector like $(\frac{1}{2}, -2, 0)$ with $(1, -4, 0)$.)
- $\text{Col}\, A$: This subspace of $\mathbb{R}^m$ is $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n\}$. The pivot columns of A form a basis for Col A.

  Here is why: To find a basis, we find the independent columns of A, so we solve $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{0}$ (or $A\mathbf{x} = \mathbf{0}$). We get rid of columns of A corresponding to free variables in these equation. If $\mathbf{a}_j$ is a column of A corresponding to the free variable x_j and we set $x_j = 1$ and all the other free variables to zero, we have a dependence relation that allows us to write $\mathbf{a}_j$ in terms of the vectors corresponding to leading variables. Then, by the Spanning Set Theorem, we can get rid this column of A. Continuing, we can get rid of all columns of A corresponding to free variables. Then, the columns of A corresponding to leading variables–*these are the pivot columns of A!!*–are a basis of $\text{Col}\, A$.
  (NB: Make sure you use the pivot columns of A and *not* those of the reduced matrix, B; in general Col $A \neq$ Col B.)