§4.2 Null Spaces, Column Spaces, and Linear Transformations

dfn: The **null space** of an $m \times n$ matrix A, denoted Nul A, is the set of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$. In set notation: Nul $A = \{\mathbf{x} \in \mathbb{R}^n | \ A\mathbf{x} = \mathbf{0}\}$

> **Theorem 2** If A is an $m \times n$ matrix, then Nul A is a subspace of $\mathbb{R}^n$, the domain of the associated linear transformation $\mathbf{x} \mapsto A\mathbf{x}$. Equivalently, the set of all solutions to a system $A\mathbf{x} = \mathbf{0}$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbb{R}^n$.

To find an explicit description of Nul A as a span of vectors, solve $A\mathbf{x} = \mathbf{0}$ and write the solution in parametric vector form. If there are no free variables in $A\mathbf{x} = \mathbf{0}$, then Nul $A = \{\mathbf{0}\}$. If there *are* free variables, successively set each free variable equal to 1 and the rest equal to 0 and this gives a linearly independent spanning set of Nul A. (The vector corresponding to free variable x_i will have a 1 in position i and every other vector in the spanning set will have a 0 in position i.) If Nul A is not just $\mathbf{0}$, the number of vectors in this spanning set for Nul A equals the number of free variables in the equation $A\mathbf{x} = \mathbf{0}$.

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dfn: The **column space** of an $m \times n$ matrix A, denoted Col A, is the set of all linear combinations of the columns of A. If $A = [\mathbf{a}_1 \, \mathbf{a}_2 \, \dots \, \mathbf{a}_n]$ then Col $A = \text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$. So, Col A is the set of all $\mathbf{b} \in \mathbb{R}^n$ such that $A\mathbf{x} = \mathbf{b}$ is consistent.

> **Theorem 3** If A is an $m \times n$ matrix, then Col A is a subspace of $\mathbb{R}^m$, the codomain of the associated linear transformation $\mathbf{x} \mapsto A\mathbf{x}$

Contrast between Nul A and Col A for an $m \times n$ matrix A.

Nul $A = \{\mathbf{x} \in \mathbb{R}^n | \ A\mathbf{x} = \mathbf{0}\}$	Col A = all linear combinations of the columns of A
1. Nul A is a subspace of $\mathbb{R}^n$	1. Col A is a subspace of $\mathbb{R}^m$.
2. Nul A is implicitly defined. Solve $A\mathbf{x} = \mathbf{0}$ to find it.	2. Col A is explicitly defined. It is Span{columns of A.}
3. It takes work to find vectors in Nul A. You must row reduce $[\,A\ \mathbf{0}\,]$ etc.	3. It is easy to find vectors in Col A. The columns of A are in Col A; other vectors in Col A are formed from them by linear combination.
4. There is no obvious relation between Nul A and the entries in A.	4. There is an obvious relation between Col A and the entries in A. Each column of A is in Col A.
5. The vector $\mathbf{u} \in \mathbb{R}^n$ is in Nul A if and only if $A\mathbf{u} = \mathbf{0}$.	5. The vector $\mathbf{b} \in \mathbb{R}^m$ is in Col A if and only if the equation $A\mathbf{x} = \mathbf{b}$ is consistent.
6. Given a specific vector $\mathbf{u}$ in $\mathbb{R}^n$, it is easy to tell if $\mathbf{u}$ is in Nul A. Just check to see if $A\mathbf{u} = \mathbf{0}$.	6. Given a specific vector $\mathbf{b}$ in $\mathbb{R}^m$, it may take time to tell if $\mathbf{b}$ is in Col A. You must row reduce $[\,A\ \mathbf{b}\,]$ to check if $A\mathbf{x} = \mathbf{b}$ is consistent.
7. Nul $A = \{\mathbf{0}\}$ if and only if $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$ (if and only if the columns of A are linearly independent).	7. Col $A = \mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^m$ (if and only if the columns of A span $\mathbb{R}^m$).
8. Nul $A = \{\mathbf{0}\}$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.	8. Col $A = \mathbb{R}^m$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ *onto* $\mathbb{R}^m$.

Note that a *function* $T : V \to W$ is a rule that assigns to each vector $\mathbf{x}$ in V a unique (meaning one and only one) vector $T(\mathbf{x})$ in W.

dfn: A **linear transformation** T from a vector space V into a vector space W, denoted $T : V \to W$, is a function (or transformation or map) from V to W satisfying the two conditions:

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in V, *(T preserves vector addition)*
2. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all $\mathbf{u}$ in V and all scalars c. *(T preserves scalar multiplication)*

- The **kernel** (or **null space**) of T is the set of all $\mathbf{u}$ in V such that $T(\mathbf{u}) = \mathbf{0}_W$. It is a subspace of V.
- The **range** of T is the set of all vectors in W of the form $T(\mathbf{x})$ for some $\mathbf{x}$ in V. It is a subspace of W.

In the case where T is a matrix transformation, $T(\mathbf{x}) = A\mathbf{x}$ for some matrix A, the kernel of T = Nul A and the range of T = Col A.