§4.1 Vector Spaces and Subspaces

A **Vector Space** is a nonempty set V of objects for which there are two defined operations, addition and scalar multiplication, subject to the ten rules below. The rules hold for all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in V and for all scalars c and d.

1. $\mathbf{u} + \mathbf{v} \in V$ *V is closed under addition.*
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ *Commutativity of +*
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ *Associativity of +*
4. $\exists\ \mathbf{0} \in V$ such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ *Existence of* $\mathbf{0}$
5. $\forall\ \mathbf{u} \in V,\ \exists\ -\mathbf{u} \in V$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ *Existence of additive inverse*
6. $c\mathbf{u} \in V$ *V is closed under scalar multiplication*
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ *Distributivity across vector addition*
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ *Distributivity across scalar multiplication*
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$ *Associativity of scalar multiplication*
10. $1\mathbf{u} = \mathbf{u}$ *Unity of* 1

Some examples of vector spaces:

- $\mathbb{R}^n$ for any positive integer n.
- $\mathbf{M}_{m\times n}$ the set of all $m \times n$ matrices.
- For any integer $n \geq 0$, define $\mathbb{P}_n$ as the set of all polynomials of degree at most (i.e. $\leq$) n:

$$\mathbb{P}_n = \{p(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n\ :\ a_0, \ldots a_n \in \mathbb{R}\}.$$

  For example, if $p(t) = 2 + 3t + t^2$, then $p \in \mathbb{P}_2$ and $p \in \mathbb{P}_3$, but $p \notin \mathbb{P}_1$ since p is a polynomial of degree two.
- C$[a, b]$ is the set of all real-valued continuous functions $\mathbf{f}(t)$ defined on the interval $[a, b]$. Define addition and scalar multiplication as follows: $\mathbf{f} + \mathbf{g}$ is the function whose value at $t \in [a, b]$ is $\mathbf{f}(t) + \mathbf{g}(t)$, and the function $c\mathbf{f}$ is the function whose value at t is $c\mathbf{f}(t)$.
- Infinite sequences $(a_0, a_1, a_2, \ldots)$ where $a_0, a_1, a_2, \ldots$ are real numbers.

Theorem If V is a vector space and $\mathbf{u} \in V$ and $c \in \mathbb{R}$, then: $0\mathbf{u} = \mathbf{0}$, $c\mathbf{0} = \mathbf{0}$ and $-\mathbf{u}$ is unique and $-\mathbf{u} = (-1)\mathbf{u}$.

NOTE: The objects in a set are called *elements* of that set. The elements of a vector space are often called *vectors*, too. As seen in the examples above, vectors can be *really* different (and cool)!

Subspaces– Subsets of vector spaces that are vector spaces themselves.

A **subspace** of a vector space V is a subset H of V that has three properties:

a. The zero vector of V is in H.
b. H is closed under vector addition. That is, for each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u} + \mathbf{v}$ is also in H.
c. H is closed under scalar multiplication. That is, for each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is also in H.

Useful tip: To check if a subset H of V is a vector space, you first see if $\mathbf{0} \in H$. If so, then you take *arbitrary* $\mathbf{u}$ and $\mathbf{v}$ in H and check if $\mathbf{u} + \mathbf{v} \in H$. Finally, you check for *arbitrary* $c \in \mathbb{R}$ and *arbitrary* $\mathbf{u} \in H$ if $c\mathbf{u} \in H$. If all these hold, then H is a subspace of V.

Facts: Every *subspace* of a vector space is itself a vector space–it is a vector space that lives inside another vector space. Note that there are *subsets* of a vector space that are not *subspaces*.

The set consisting of only the zero vector in any vector space V is a subspace of V. It is called the **zero subspace** and written as $\{\mathbf{0}\}$.

Theorem 1 If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ are vectors in a vector space V, then span$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is a subspace of V.

The set span$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is called the **subspace spanned** by $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$. Given any subspace H of V, a **spanning set** for H is a set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ such that $H = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$.