Proof Techniques

Let \( P \) and \( Q \) be statements. (For example, \( P \) could be a statement like “\( u \) and \( v \) are linearly independent.”)

If/Then Statements: “If \( P \), then \( Q \)” means that whenever \( P \) is true, \( Q \) then must be true. To prove this directly, assume \( P \) is true and use this and other properties to prove that \( Q \) is true.

Abbreviation: “\( P \implies Q \)”, which is pronounced “\( P \) implies \( Q \)”. 

If and only if statements: “\( P \) if and only if \( Q \)” means both “if \( P \), then \( Q \)” and “if \( Q \), then \( P \)”. To prove an if and only if statement, prove both \( P \implies Q \) and \( Q \implies P \).

Abbreviations: we can also write “\( P \) if and only if \( Q \)” as “\( P \iff Q \)” or as “\( P \iff Q \)”.

Negation: The negation of \( P \) is “not \( P \).”

This is not always as simple as it sounds, and you should be very careful to avoid mistakes when taking negations.

For example, the negation of the statement “\( c_1, c_2, \ldots, c_n \) are all zero” is NOT “\( c_1, c_2, \ldots, c_n \) are all nonzero”. Rather, the negation of the statement “\( c_1, c_2, \ldots, c_n \) are all zero” is “at least one of \( c_1, c_2, \ldots, c_n \) is nonzero”.

Abbreviation: \( \neg P \)

Contrapositive: The contrapositive of “if \( P \), then \( Q \)” is “if not \( Q \), then not \( P \)” (\( \neg Q \implies \neg P \)).

The contrapositive of an if/then statement is logically equivalent to the original if/then statement. This means that another way to prove \( P \implies Q \) is to prove \( \neg Q \implies \neg P \).

More on Negation: The negation of “if \( P \), then \( Q \)” is “\( P \) and \( \neg Q \)”. We use this for proof by contradiction and for counterexamples, below.

Counterexamples: To show that a statement is false, you just need one example where it is false, which we call a counterexample. In particular, to show that an if/then statement “if \( P \), then \( Q \)” is false, a counterexample would be any example where we have “\( P \) and \( \neg Q \)”, (i.e, an example in which \( P \) is true but \( Q \) is false).

Proof by Contradiction for If/Then Statements: Another way to prove “if \( P \), then \( Q \)” is to assume “\( P \) and \( \neg Q \)” and derive a contradiction.

We would typically write this by first saying “assume \( P \)” and then saying “assume \( \neg Q \)” and then doing some work to derive a contradiction. You should be very careful with this method, as it is easy to make mistakes with what exactly “\( \neg Q \)” is, and what qualifies as a contradiction.

In many situations, it is more straightforward to write a direct proof or prove the contrapositive, \( \neg Q \implies \neg P \), than to do a proof by contradiction.